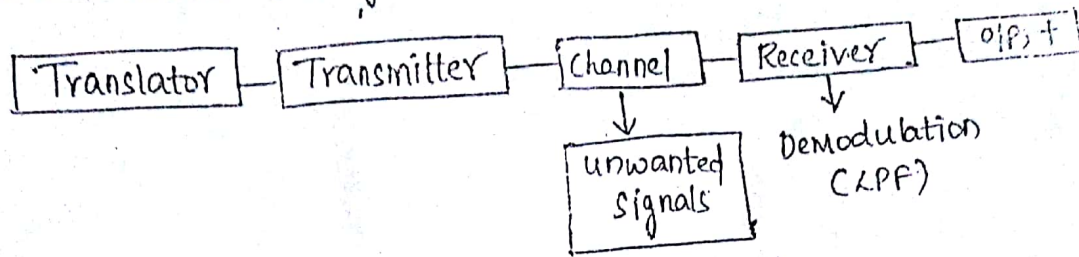
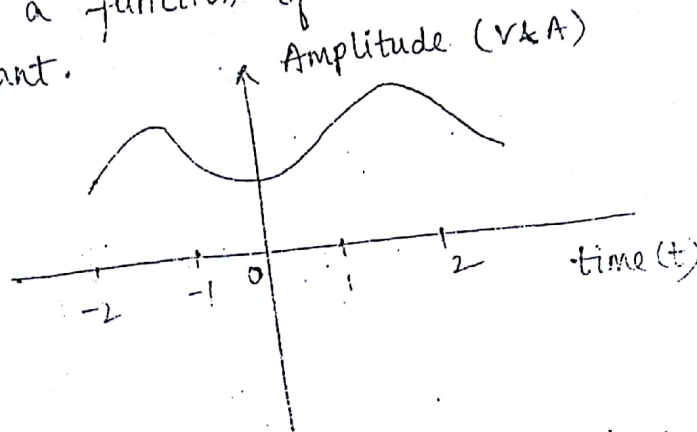


Signals & Systems

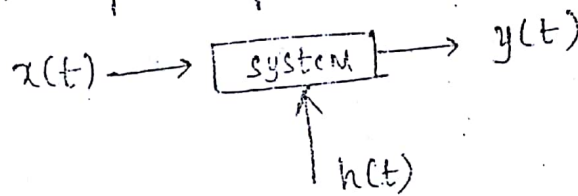
Telecommunication: long distance communication input raw signal



Signal: Signal is a function of time which conveys information at any instant.



System: A system is a physical parameter which maps the relation b/w input response and output as shown below.



$$y(t) = T[x(t)]$$

$$y(t) = x(t) * h(t)$$

* $\rightarrow$ convolution

It is a process of obtaining system response by continuous mixing of input response with its impulse response.

$x(t) \rightarrow$ i/p response

$h(t) \rightarrow$ impulse response

$y(t) \rightarrow$ o/p response

Basic signal classification

Continuous time
or $x(t)$

Analog signals
t-continuous
time.

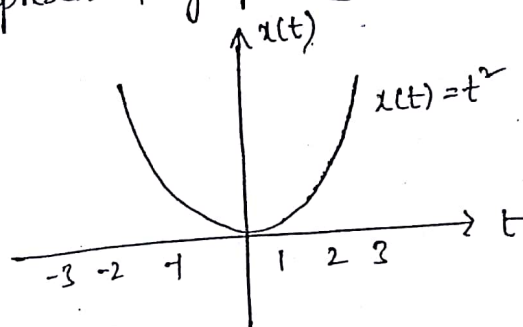
Discontinuous time
or $x(n)$

Discrete signals
n-discontinuous
time

Digital
signals

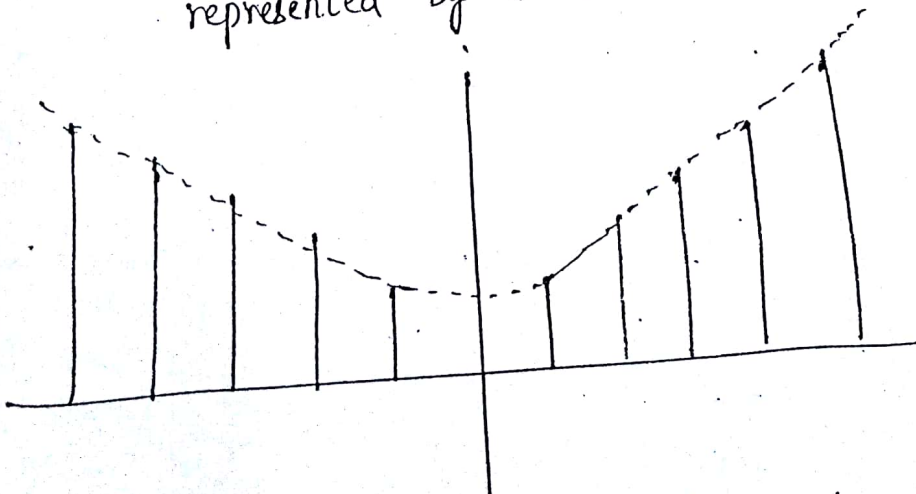
Continuous time or Analog signal:

A signal which is continuous both in 'T' and 'Amp' (Time period)
These are represented graphically in terms of t vs $x(t)$.



Discontinuous time or Discrete signals:

The signal which are continuous in 'amp' & discrete
in time period
represented by 'n' vs $x(n)$.



discrete slope or signal
't' continuous.

Note: (i) The arrow in the above signal indicates origin.

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(ii) If arrow isn't indicated in above signal then 1st sample is itself at original signals are known as right sided signals.

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* U₁

Digital signal

A signal discrete in both the forms i.e. 'Amp' & 'time period' are known as digital signal.

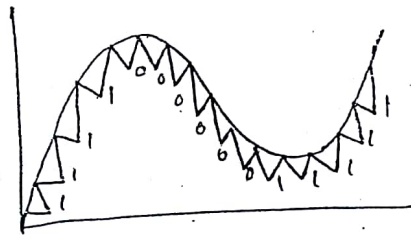
A continuous time signal is converted to digital signal by quantisation process.

Quantisation:

This is the process of uniform staircase approximation in which the difference in their quantised levels ($\delta_1, \delta_2, \delta_3$) are to be considered. If the difference is +ve value the signal is assumed to be ON or high.

If its -ve value, then signal is off or low as

shown below



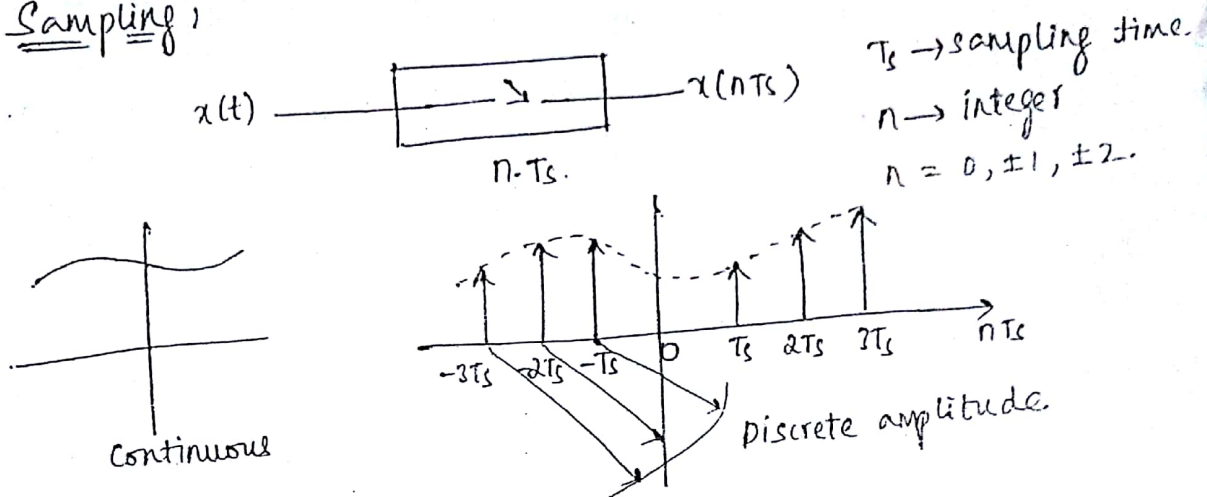
if $\delta_2 - \delta_1 = +ve$ (ON)
 $\delta_2 - \delta_1 = -ve$ (OFF)

|| || 00000 || ||| (unipolar Q)

|| || -+-+--+ || ||| (bipolar Q)

Note! A continuous time signal is converted to discrete time signal the process is called sampling.
 * a continuous time signal $\rightarrow$ digital signal by process of quantisation.

Sampling:



Q1 A continuous time signal is given $2\cos(150\pi t - 60^\circ)$ degrees in continuous. this is sampled at frequency of 200KHz .

$$x(n) = ?$$

$$T_s = 0.5 \times 10^{-3} = \frac{1}{200}$$

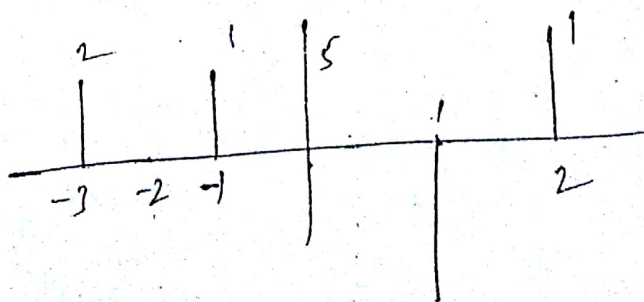
$$t \rightarrow 5\text{ms}$$

$$x(nT_s) = 2\cos\left(150\pi \times 0.0005 - \frac{\pi}{3}\right)$$

$$x(nT_s) = 2\cos\left(\frac{2\pi n}{4} - \pi/3\right), \text{ in radians in discrete time}$$

Q: Represent the discrete signal graphically

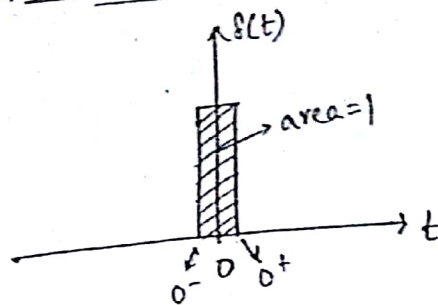
$$x(n) = \{2, 0, 1, 5, -2, 1\}$$



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The standard representation of signals:

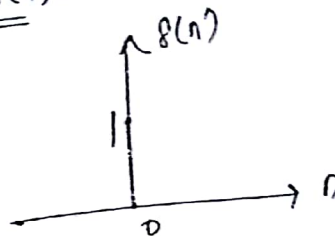
* Unit impulse $\delta(t)$:



$$\int_{0^-}^{0^+} \delta(t) \cdot dt = 1 ; t=0$$

$$= 0 ; t \neq 0$$

* Unit impulse $\delta(n)$:



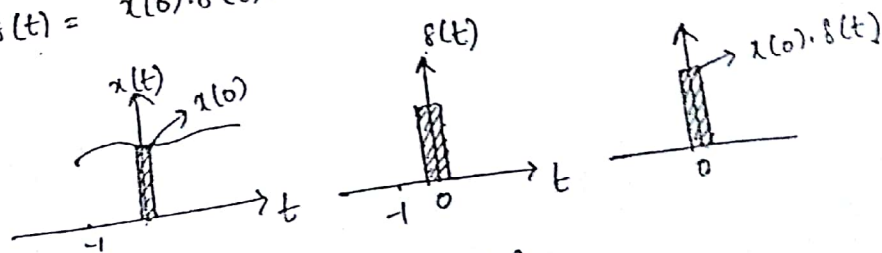
$$\delta(n) = 1 ; n=0$$

$$= 0 ; n \neq 0$$

(1) & (2) $\rightarrow$ sampling properties

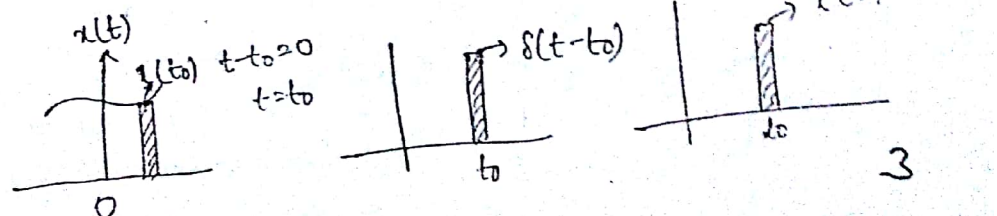
$\rightarrow$ Properties of $\delta(t)$:

(1) $x(t) \cdot \delta(t) = x(0) \cdot \delta(t)$



(2) $\int_{-\infty}^{\infty} x(t) \cdot \delta(t) \cdot dt = \int_{-\infty}^{\infty} x(0) \cdot \delta(t) \cdot dt = x(0)$

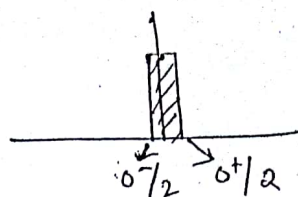
(3) $x(t) \cdot \delta(t-t_0) = x(t_0) \delta(t-t_0)$



$$(4) \int_{-\infty}^{\infty} x(t) \cdot \delta(t-t_0) dt = \int_{-\infty}^{\infty} x(t_0) \delta(t-t_0) dt = x(t_0).$$

(5) Scaling

$$\delta(at) = \frac{1}{|a|} \delta(t)$$



(6) Convolution

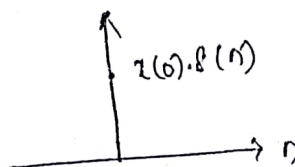
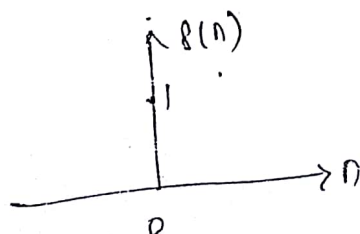
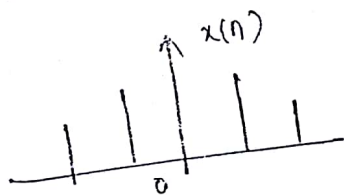
$$(1) x(t) * \delta(t) = x(t)$$

$$(2) x(t) * \delta(t-t_0) = x(t-t_0) * \delta(t) = x(t-t_0)$$

$$(3) x(t-t_1) * \delta(t-t_2) = x(t-t_1-t_2)$$

Property for $\delta(n)$

$$(1) x(n) \cdot \delta(n) = x(0) \cdot \delta(n)$$



In discrete signals
signal strength
depends on
summation

$$(2) \sum_{n=-\infty}^{\infty} x(n) \cdot \delta(n) = \sum_{n=-\infty}^{\infty} x(0) \delta(n) = x(0)$$

C.T	D.T
t	n
T	N
$\int$	$\sum$
$\frac{d}{dt}$	D

crete
nolds
strength
ends on
information

* Evaluate the following

- (1) $x(n) * \delta(n) = x(n)$
- (2) $x(n) * \delta(n-n_0) = x(n-n_0)$
- (3) $x(n) * \delta(n-n_0) = x(n-n_0)$
- (4) $\sum_{n=-\infty}^{\infty} x(n) \cdot \delta(n-n_0) = x(n_0)$
- (5) scaling $\delta(n) = \delta(n)$ [scaling is unaffected]
- (6) convolution
 1. $x(n) * \delta(n) = x(n)$
 2. $x(n) * \delta(n-n_0) = x(n-n_0)$
 3. $x(n-n_1) * \delta(n-n_2) = x(n-n_1-n_2)$

$$1. \int_{-\infty}^{\infty} e^{-x} \delta(t-3) dt$$

$\downarrow$
 $t=3$

$$x(t) \cdot t-3=0$$

$$x(3) = e^{-9}$$

$$2. \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

$$3. \int_{-\infty}^{\infty} e^{-j2t} \delta(t) dt = \frac{1}{2} [x(0)] = \frac{1}{2}$$

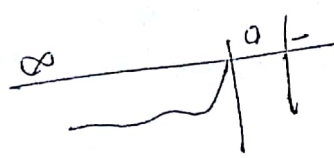
$$4. \int_{-\infty}^{\infty} (t+1) \delta(t+1) dt = 2$$

$$5. \sum_{n=-\infty}^{\infty} e^{2n} \delta(n+2) = e^{-4}$$

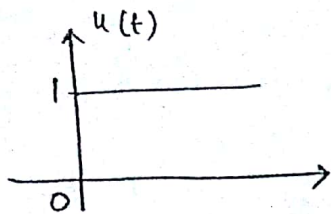
$$6. \sum_{n=0}^{\infty} n \delta(n+1) = 0$$

$$7. \sum_{n=2}^{\infty} (n-1) \delta(n-2) = 8$$

$$8. \sum_{n=-\infty}^{\infty} \delta(2n) = 1$$



⇒ Unit step signal: $[u(t)]$:



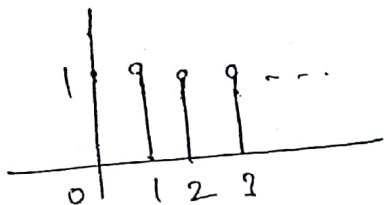
$$u(t) = 1 ; t > 0$$

$$= 0 ; t < 0$$

$$u(0^+) = 1$$

$$u(0^-) = 0$$

unit step $[u(n)]$:



$$u(n) = 1 ; n \geq 0$$

$$= 0 ; n < 0$$

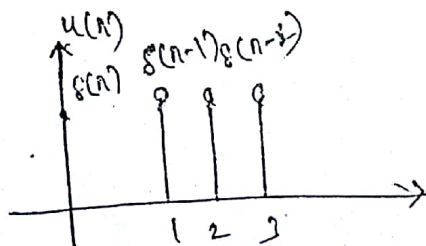
* Properties:

(1) $\frac{d}{dt}[u(t)] = \delta(t)$

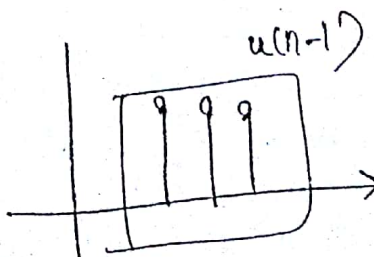
→ $u(t) = \int_0^{\infty} \delta(t) \cdot dt$ → signal consisting finite strength

(2) For $u(n)$

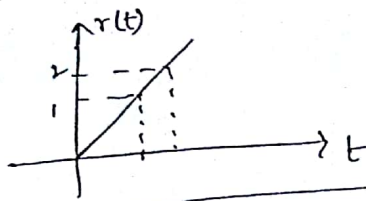
(1) $u(n) = \sum_{k=0}^{\infty} \delta(n-k)$



(2) $\delta(n) = u(n) - u(n-1)$



⇒ Unit Ramp function $[r(t)]$:



$$\begin{aligned} r(t) &= t ; t > 0 \\ &= 0 ; t < 0. \end{aligned}$$

Properties:

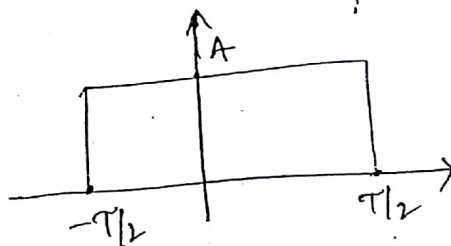
1. $\frac{d}{dt} r(t) = u(t)$

2. $\frac{d^2}{dt^2} r(t) = \delta(t)$

3. $r(t) = \int_0^{\infty} u(t) \cdot dt$

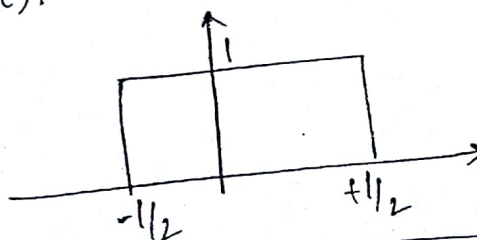
⇒ Gate Function or Rectangular function:

↳ $A\pi\left(\frac{t}{\tau}\right)$ ↳ $A\text{rect}\left(\frac{t}{\tau}\right)$



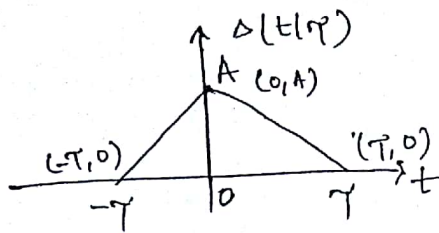
$$A\pi\left(\frac{t}{\tau}\right) = A ; -\tau/2 < t < \tau/2$$

→ Sketch $\pi(t)$



$$\pi(t) = 1 ; -\frac{1}{2} < t < \frac{1}{2}$$

⇒ Triangular function: $[A \Delta(t/\tau)]$:



symmetrical
on y-axis

$$A \Delta(t/\tau)$$

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$$

$$\begin{matrix} (-\tau, 0) & (0, A) \\ x_1, y_1 & x_2, y_2 \end{matrix}$$

$$\frac{\Delta(t/\tau) - 0}{t + \tau} = \frac{A - 0}{0 + \tau}$$

$$x = t, y = \Delta(t/\tau)$$

⇒ Sin

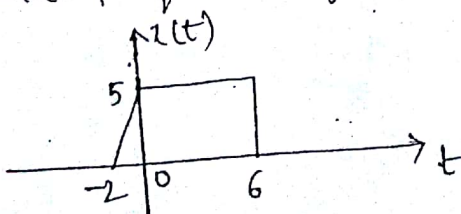
$$\frac{\Delta(t/\tau)}{t + \tau} = \frac{A}{\tau}$$

$$\Delta(t/\tau) = \frac{A}{\tau} (t + \tau)$$

$$\boxed{\begin{aligned} \Delta(t/\tau) &= A(1 + t/\tau) ; -\tau < t < 0 \\ &= A(1 - t/\tau) ; 0 < t < \tau \end{aligned}}$$

$$\Rightarrow \Delta(t/\tau) = A \left[1 - \frac{|t|}{\tau} \right] ; |t| < \tau$$

Problem:
Obtain the standard equation of the signal



$$(-2, 0), (0, 5)$$

$$x_1, y_1 \quad x_2, y_2$$

$$(x, y) = (t, x(t))$$

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$$

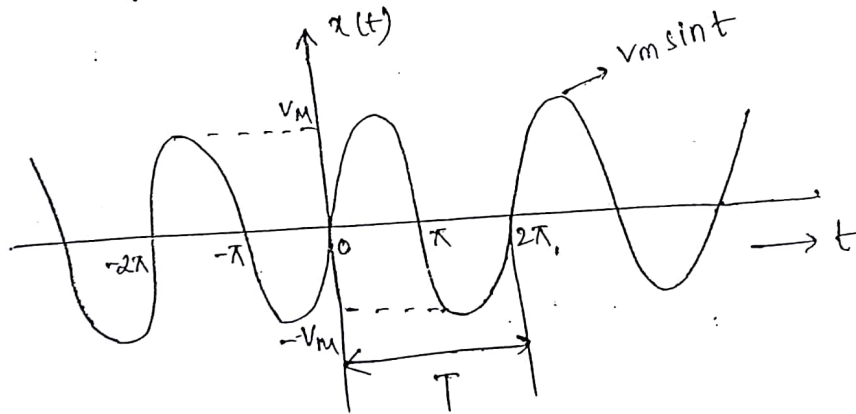
$$\frac{x(t)-0}{t+2} = \frac{5-0}{0+2}$$

$$x(t) = \frac{5}{2}(t+2)$$

$$-2 < t < 0$$

$$= 5; 0 < t < 6$$

⇒ Sinusoidal Signals [sinusoids]



$$T = 2\pi, \omega = 2\pi f = \frac{2\pi}{T}$$

$$\omega = \frac{2\pi}{2\pi} = 1 \text{ rad/s}$$

$$x(t) = A \sin(\omega t + \theta)$$

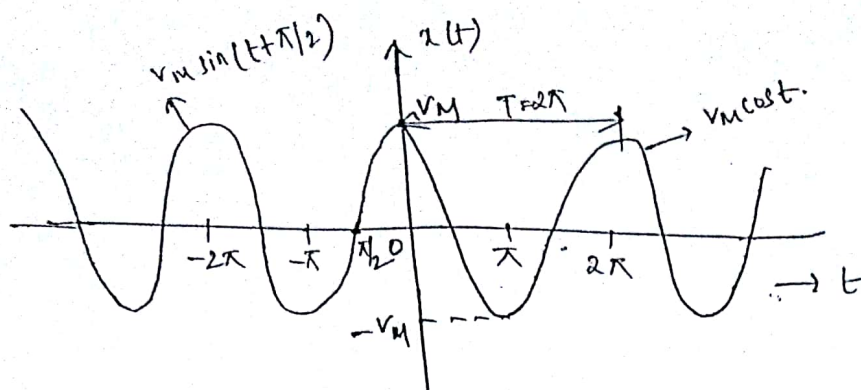
$A \rightarrow$ max. amplitude (V or A)

$\omega \rightarrow$ angular frequency (rad/sec)

$\theta \rightarrow$ phase (degrees / radian)

↓ cont. signal ↓ dis. signal

Cosine

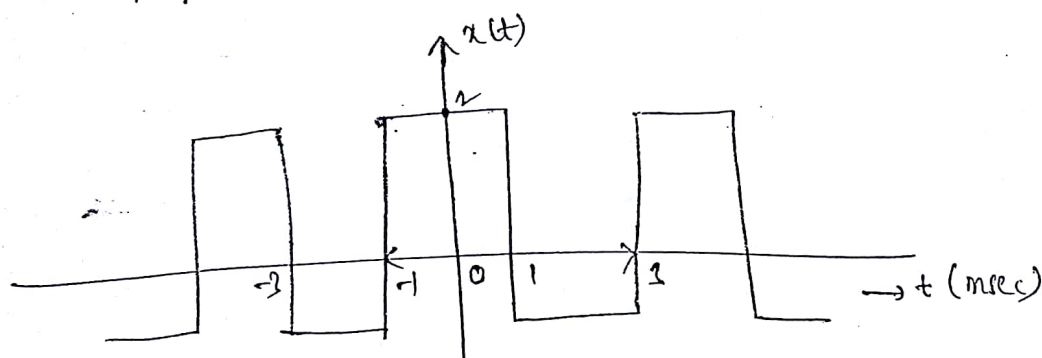


$$x(t) = A \cos(\omega t + \theta)$$

$$T = 2\pi$$

$$\omega = \frac{2\pi}{2\pi} = 1 \text{ rad/s}$$

→ For the given periodic signals find their given fundamental periods, frequencies in rad/sec & Hz and eqn. of signal.



$$T = [h \cdot L - L \cdot L] \text{ cycle}$$

$$= [3 - (-1)]$$

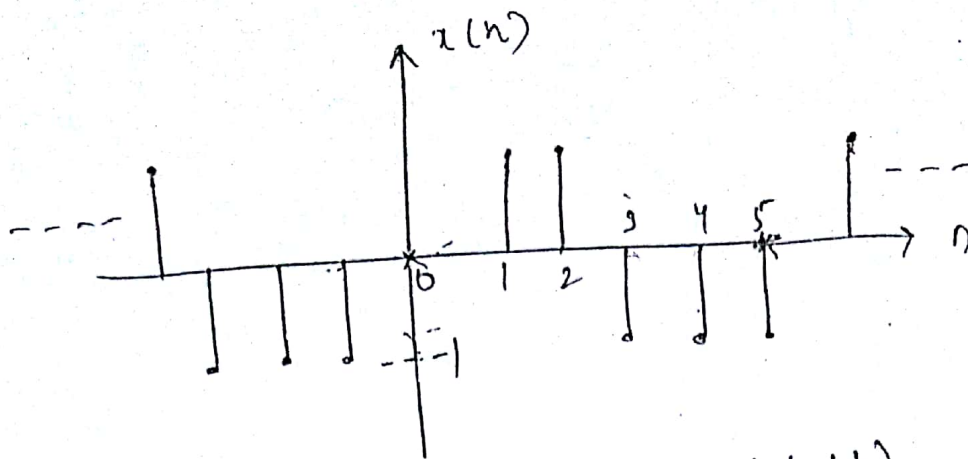
$$= 4 \text{ msec}$$

$$f = \frac{1}{T} = \frac{1}{4 \text{ ms}} = 250 \text{ Hz}$$

$$\omega = 2\pi f = 2\pi \times 250 = 500\pi \text{ rad/s}$$

$$\boxed{x(t) = 2; \quad -1 < t < 1}$$
$$= -1; \quad 1 < t < 3$$

* Ex!



$$N = (U.L - L.L + 1)$$

$$= (5 - 0 + 1) = 6 \text{ sec.}$$

$$f_0 = \frac{1}{N} = \frac{1}{6} \text{ samples/sec}$$

$$\omega_0 = 2\pi \times \frac{1}{6} = \frac{\pi}{3} \text{ rad/s.}$$

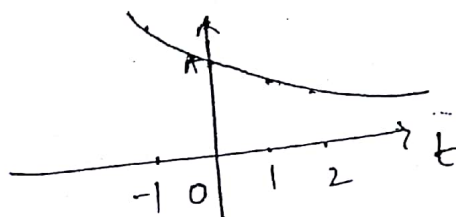
* Exponential Signals

→ Continuous exponentials:

$$x(t) = Ae^{-at}$$

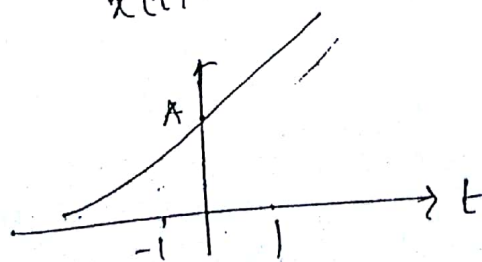
Case(1): $a > 0$ [exponentially decreasing signal]

$$a = 1, x(t) = Ae^{-t}$$



Case(2): $a < 0$ [↑]

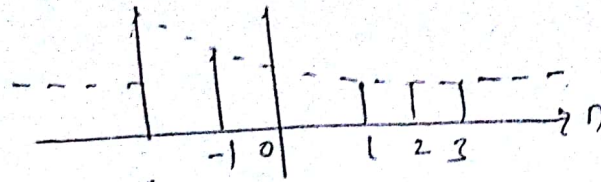
$$x(t) = Ae^{at}$$



→ Discrete exponents:

$$x(n) = a^n$$

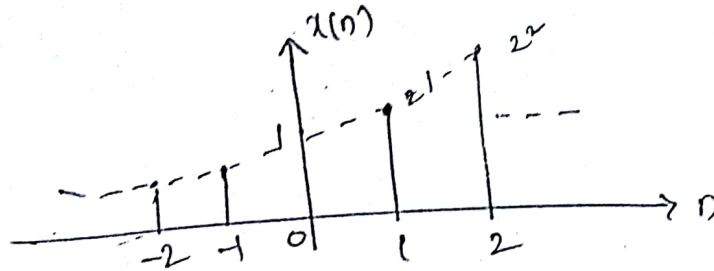
Case(1): $0 < a < 1$ [↓]
 $a = 0.2, x(n) = a^{0.2}$



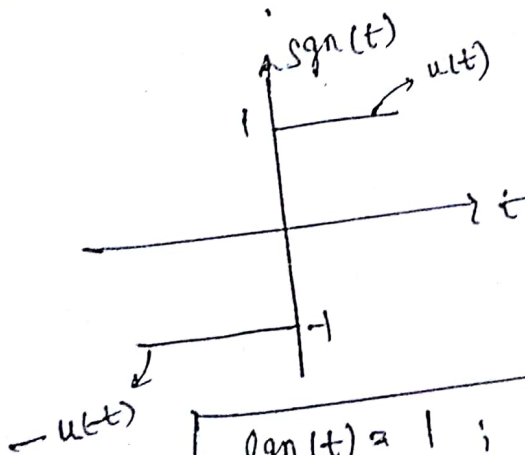
Case (2)

$(a > 1) \uparrow$

$$a = 2; \quad x(n) = 2^n$$



$\Rightarrow$ Signum function $[sgn(t)]$



$$\begin{aligned} sgn(t) &= 1; & t > 0 \\ &= -1; & t < 0 \end{aligned}$$

$$* \quad sgn(t) = u(t) - u(-t)$$

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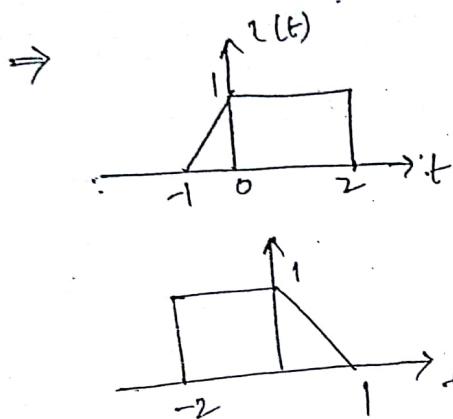
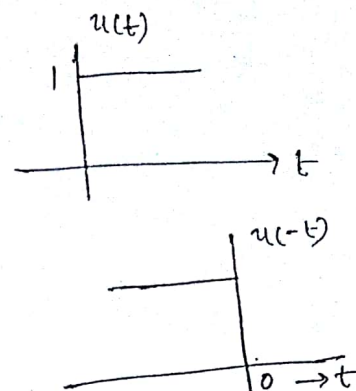
Basic operations on signals:

Time Reversal
(folding)

$$t \leftrightarrow -t$$

$$x(t) \leftrightarrow x(-t)$$

$$x(n) \leftrightarrow x(-n)$$



$$\Rightarrow x(n) = \{-3, 0, 1, 5, 2\}$$

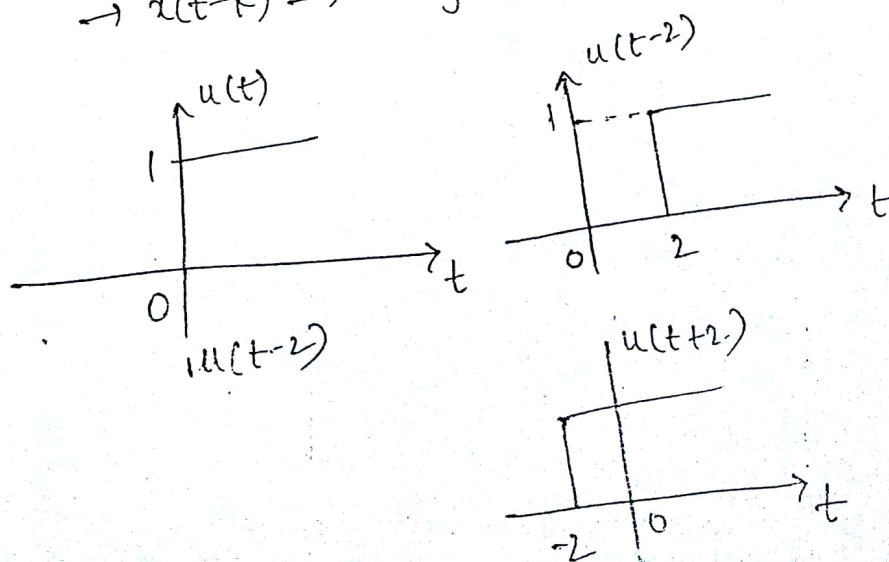
$$x(-n) = \{2, 5, 1, 0, -3\}$$

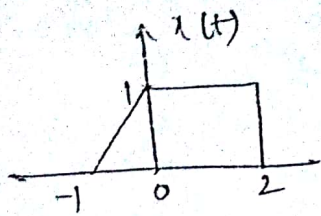
* Time Shifting

→ A signal is said to be time shifted if it is either delayed in time or advanced in time.

$$x(t) \rightarrow x(t+k) \rightarrow \text{advanced in time.}$$

$$\rightarrow x(t-k) \rightarrow \text{delayed in time}$$





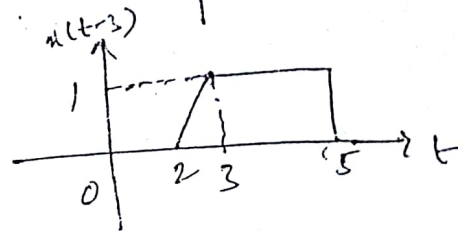
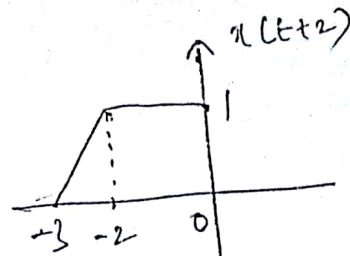
$$[-1+3, 0+3, 2+3]$$

$$[2, 3, 5]$$

$$x(t+2), x(t-3)$$

$$[-1-2, 0-2, 2-2]$$

$$[-3, -2, 0]$$



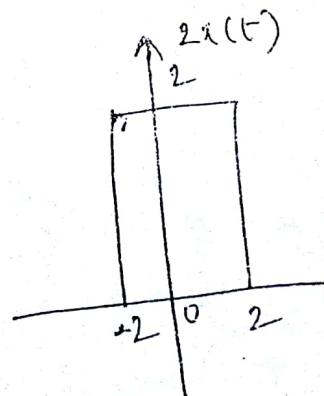
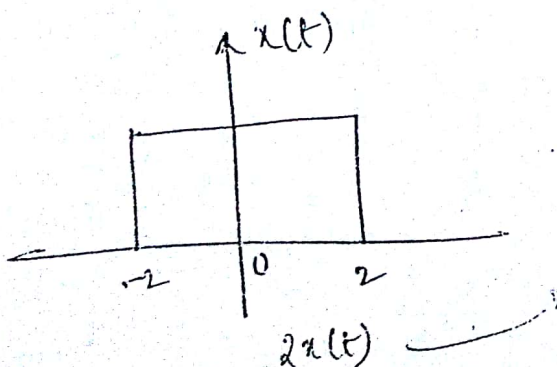
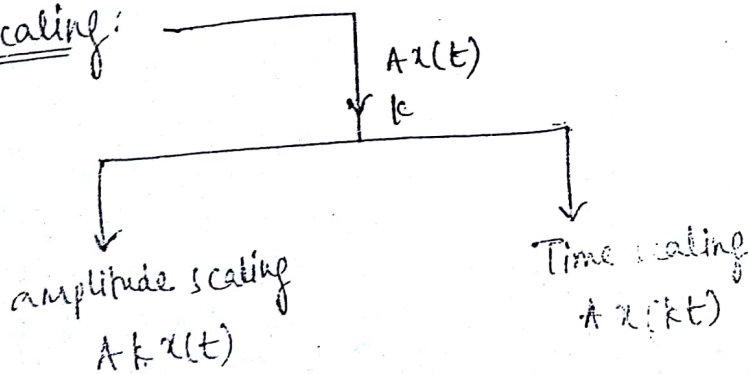
Sketch

$$x(n) = \{-3, 0, \underset{\uparrow}{5}, 1, 2, -3\}$$

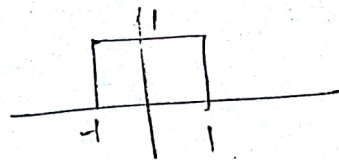
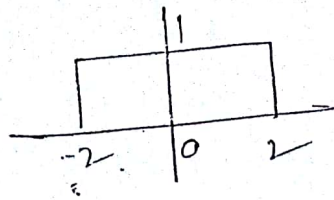
$$x(n-2) = \{-3, 0, \underset{\uparrow}{5}, 1, 2, -3\}$$

$$x(n+1) = \{-3, 0, \underset{\uparrow}{5}, 1, 2, -3\}$$

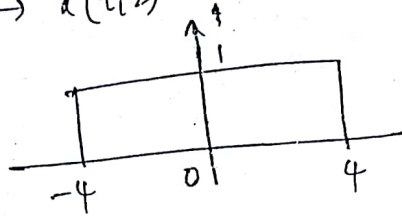
Scaling:



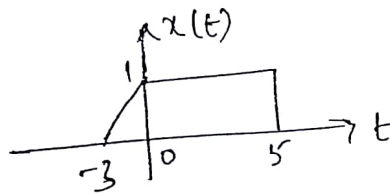
Sketch



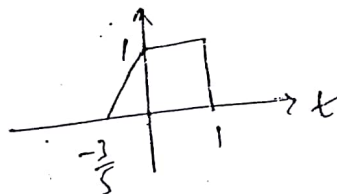
$\rightarrow x(t/2)$



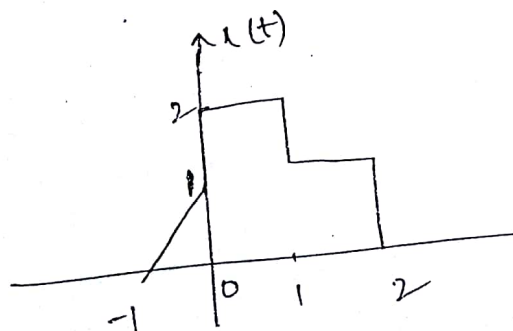
Sketch:



$x(5t)$



Sketch the following on the signal $x(t)$:



1. $x(2t+1)$

2. $x(2t-1)$

3. $x(1-t)$

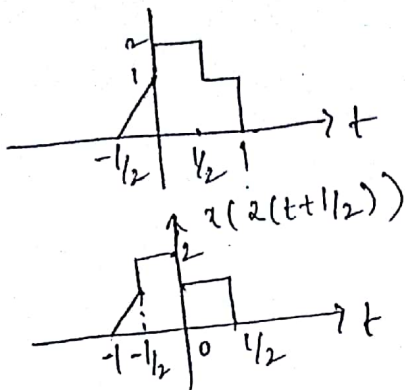
4. $x(1-2t)$

5. $x(2-3t)$

(1)

$$x(2t+1) = x\left[2\left(t+\frac{1}{2}\right)\right]$$

$x(2t)$

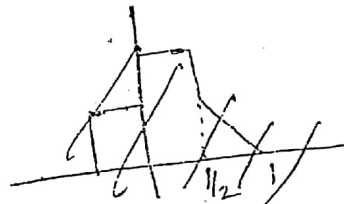
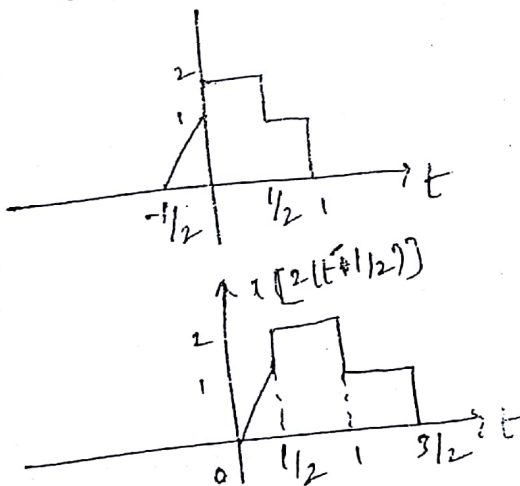


$$\begin{aligned} & \frac{-1}{2} \rightarrow \frac{-1}{2} \\ & \frac{1}{2} \rightarrow \frac{1}{2} \\ & \frac{1}{2} - \frac{1}{2} = 0 \\ & 0 - \frac{1}{2} = -\frac{1}{2} \\ & 1 - \frac{1}{2} = \frac{1}{2} \end{aligned}$$

(4)

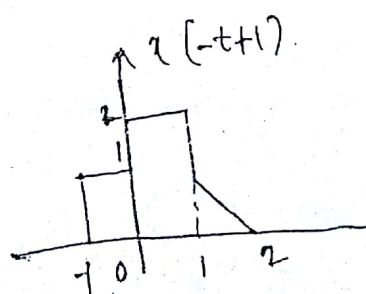
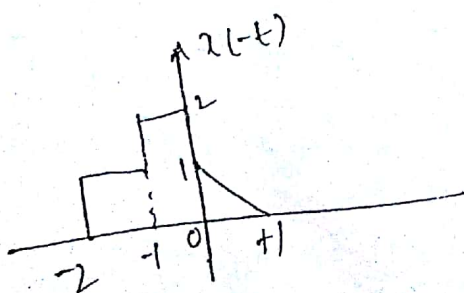
(2) $x(2t-1) = x\left[2\left(t-\frac{1}{2}\right)\right]$

$x(2t)$



$$\begin{aligned} & \frac{-1}{2} \rightarrow \frac{-1}{2} \\ & \frac{1}{2} \rightarrow \frac{1}{2} \\ & \frac{1}{2} - \frac{1}{2} = 0 \\ & 0 - \frac{1}{2} = -\frac{1}{2} \\ & 1 - \frac{1}{2} = \frac{1}{2} \end{aligned}$$

(3) $x(1-t) = x(-t+1)$
 $= x[-(t-1)]$



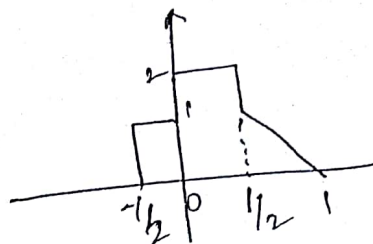
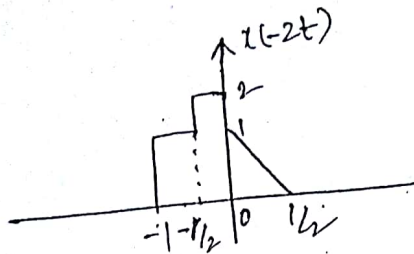
$$\begin{aligned} & 2-t \\ & -1 \\ & -1-t \\ & 0 \\ & 0-t \\ & -1-t \\ & -1-t \end{aligned}$$

$$(4) \quad x(1-2t)$$

$$\Rightarrow x(-2t+1)$$

$$\Rightarrow x(-(2t-1)) \Rightarrow x(-2(t-1/2))$$

$$\begin{aligned} -1 + 1/2 &= -1/2 \\ -1/2 & \\ 1/2 + 1/2 &= 1 \\ 0 & \end{aligned}$$

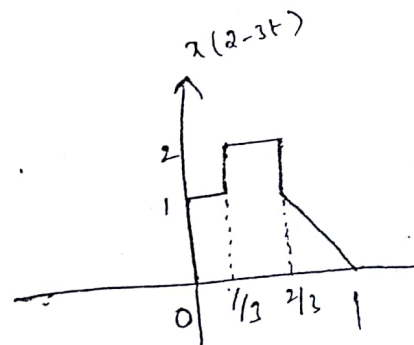
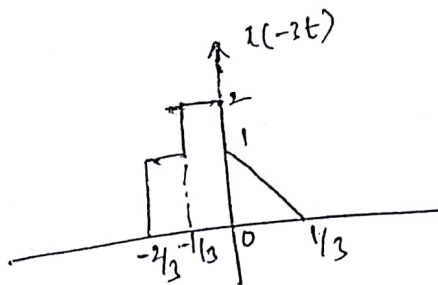


$$(5) \quad x(2-3t)$$

$$\Rightarrow x(-3t+2)$$

$$\Rightarrow x(-3(t-2/3))$$

$$\begin{aligned} -2/3 + 2/3 &= 0 \\ -1/3 + 2/3 &= 1/3 \\ 0 + 2/3 &= 2/3 \\ 1/3 + 2/3 &= 1 \end{aligned}$$



$$\Rightarrow x(n) = \{-3, 2, 5, 7, 6, -2, 1\}$$

$$1) \quad x(2n) = \{2, 7, 6, 1\}$$

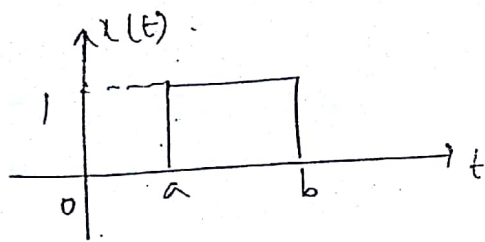
$$2) \quad x(1+n) = \{1, -2, 6, 7, 5, 2, -3\}$$

$$3) \quad x(1-2n) = \{-2, 7, 5, -3\}$$

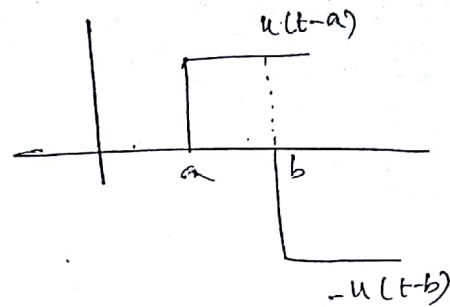
$$\begin{aligned} 1-2 \times 2 &= -3 \\ -3 & \end{aligned}$$

→ Waveform Synthesis

signal construction using unit-step function:

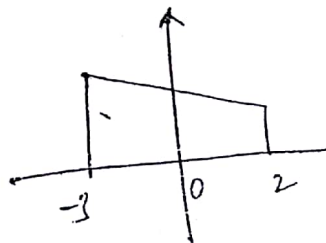
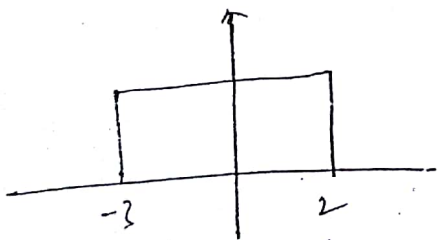
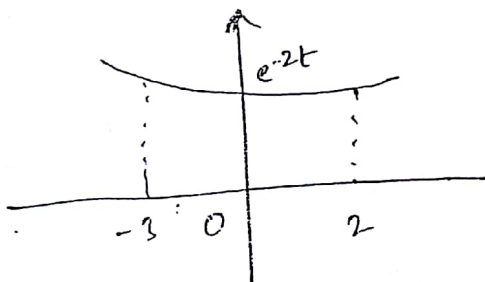


$$x(t) = 1, \quad a < t < b.$$



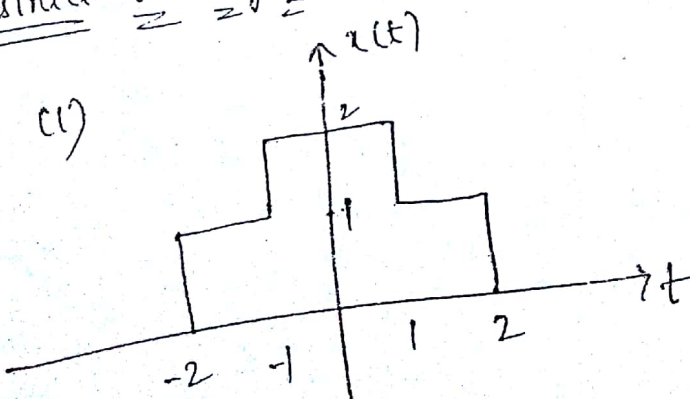
$$x(t) = u(t-a) - u(t-b)$$

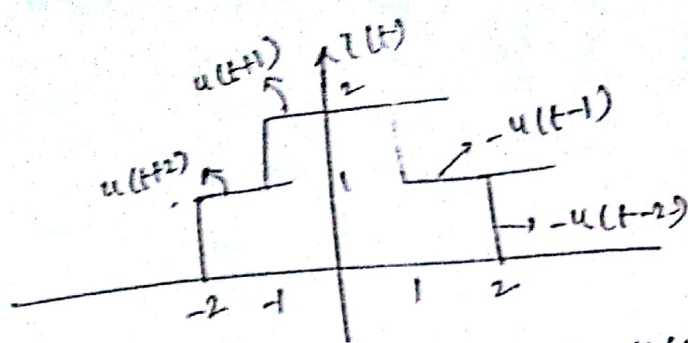
$$x(t) = e^{-2t} [u(t+3) - u(t-2)]$$



Construct the signal

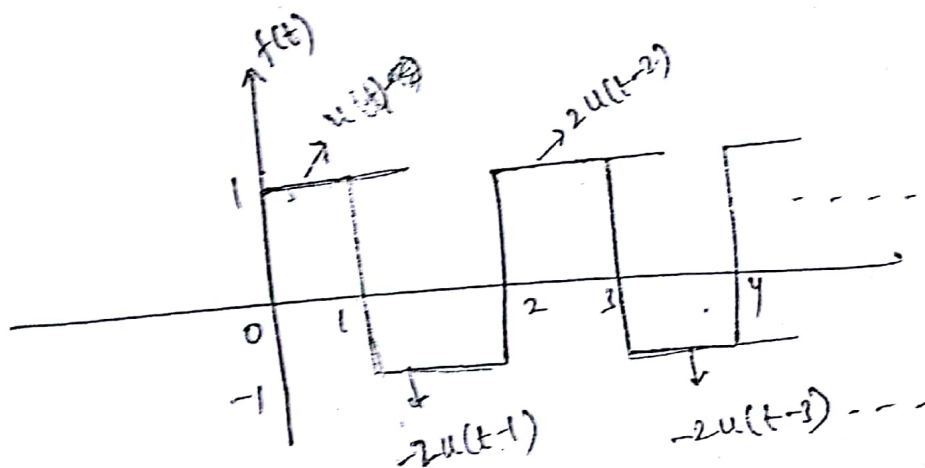
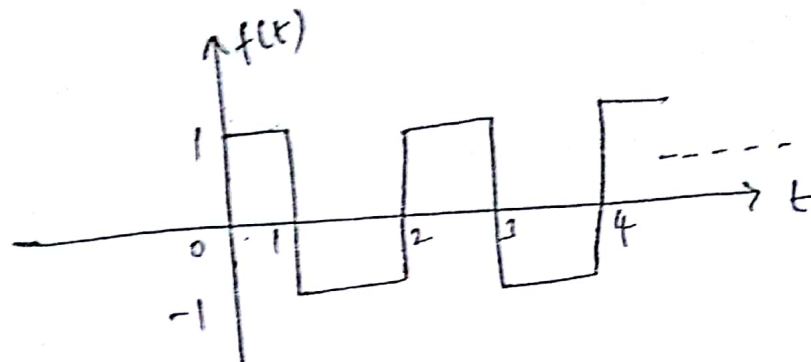
(1)





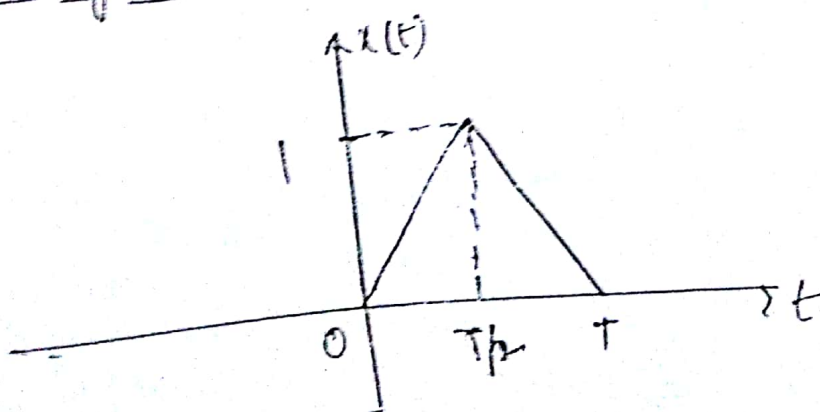
$$x(t) = u(t+2) + u(t+1) - u(t-1) - u(t-2)$$

Q2)



$$f(t) = u(t) - 2u(t-1) + 2u(t-2) - 2u(t-3) \dots$$

* Wave form synthesis using Ramp functions:

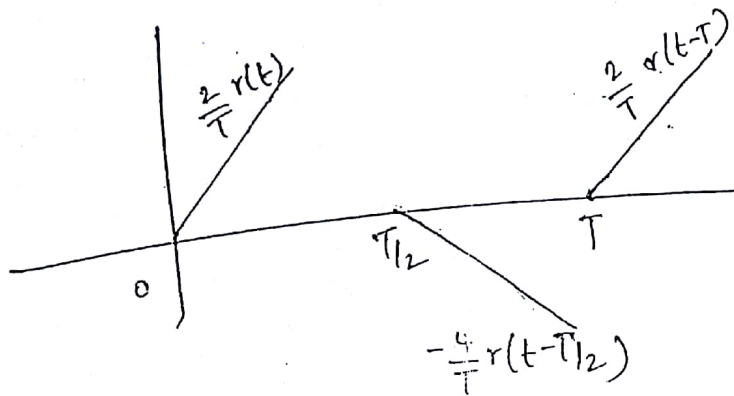
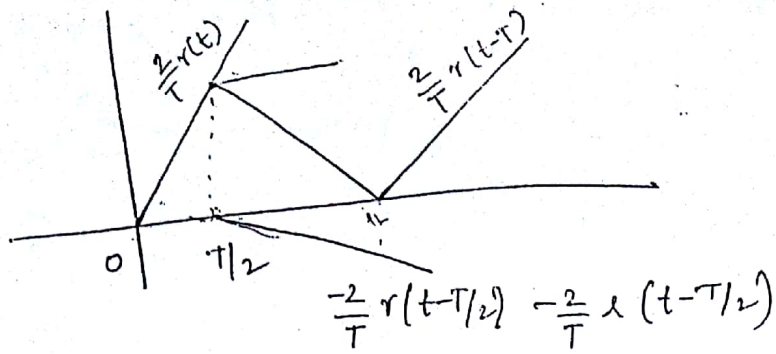


(0,0) (T/2,1)

(T,0)

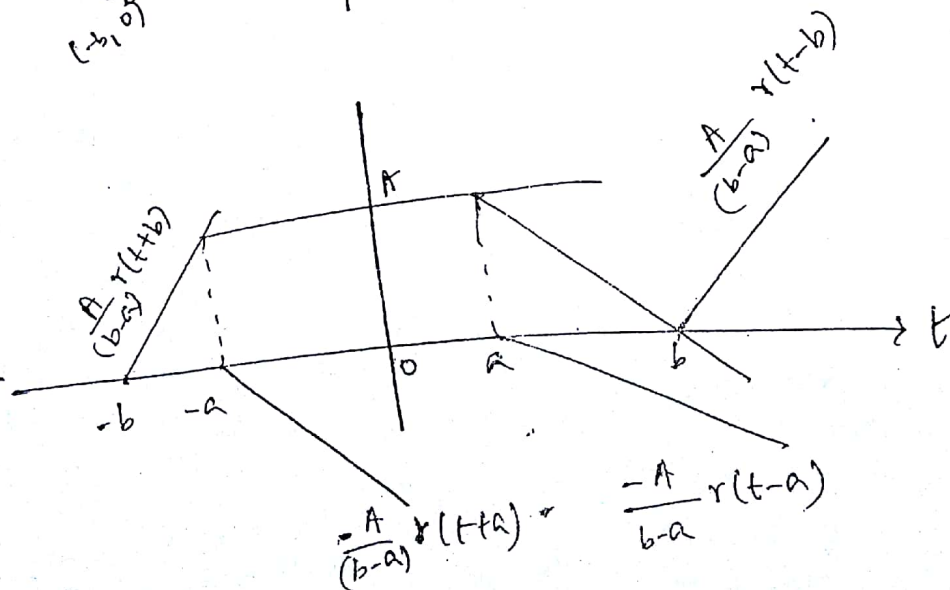
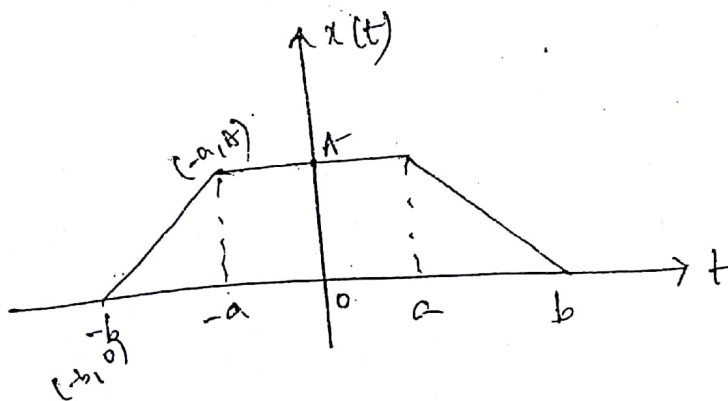
$\frac{0-T}{T/2-0} = \frac{0-1}{T/2-0}$



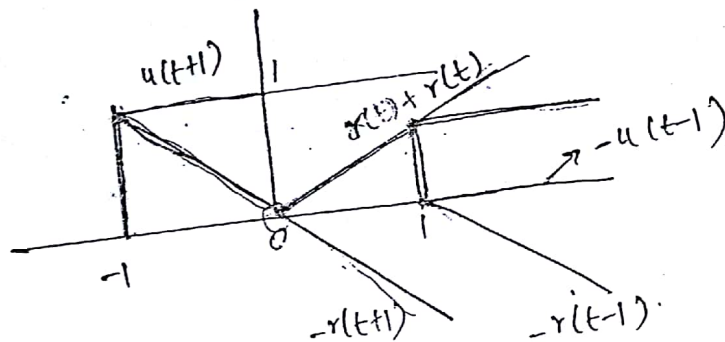
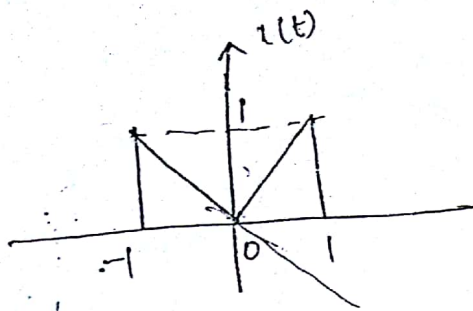


$$\Rightarrow \frac{-2}{T}r(t - T/2) - \frac{4}{T}r(t - T) + \frac{2}{T}r(t - T)$$

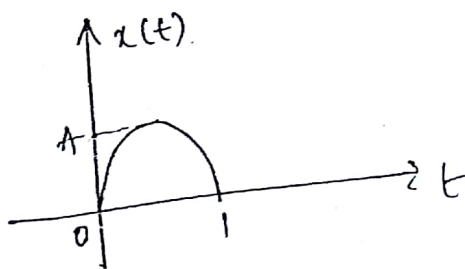
⇒



Wla

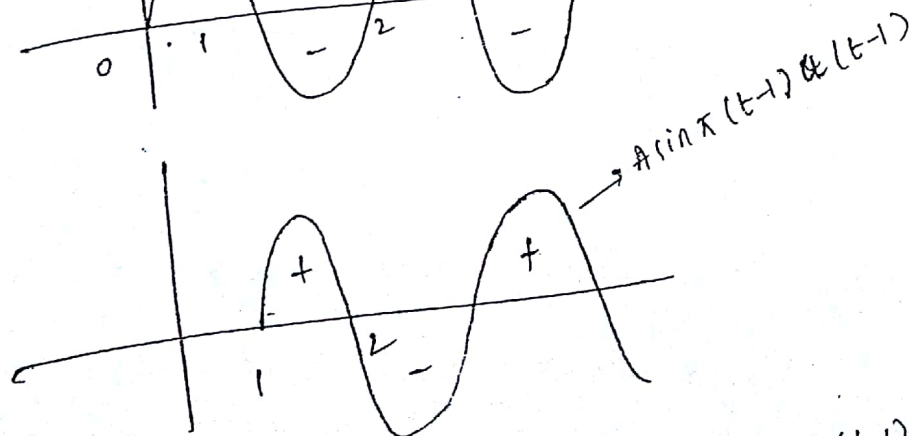
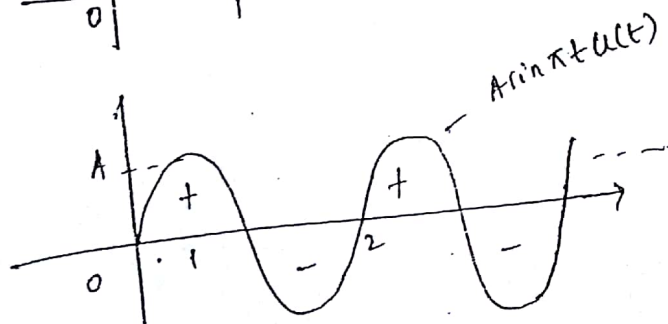


Wave forms using sinusoids

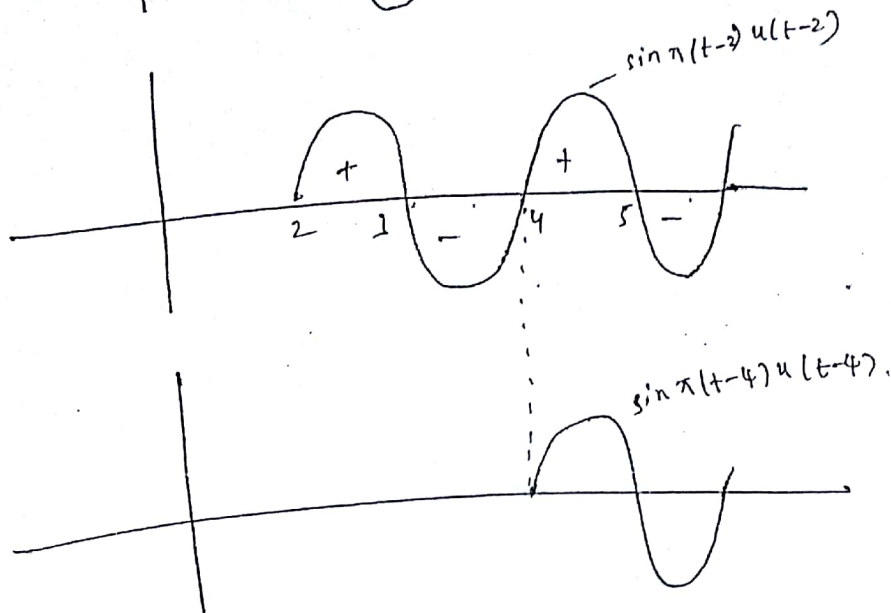
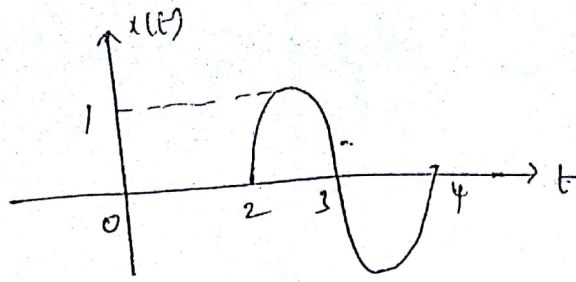


$$T=2$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi \text{ rad/s}$$



$$x(t) = A \sin \frac{\pi t}{2} u(t) + A \sin \frac{\pi(t-1)}{2} u(t-1)$$

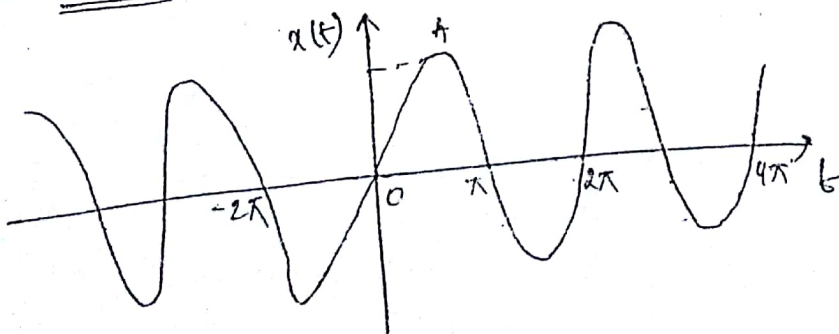


10/11/13

Complete classification of signals:

→ Common classification to both continuous time and discrete time signals.

(1) Periodic & non-periodic signals;



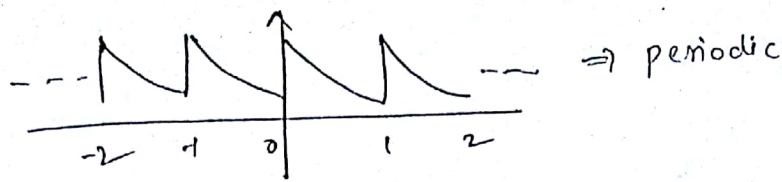
→ Any signal which repeats with a particular time period (fundamental period) are known as periodic signals. For a signal to be periodic → $x(t) = x(t \pm kT)$
 → $x(n) = x(n \pm kN)$

Sem

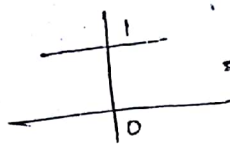
Find
indi

$k \rightarrow 0, \pm 1, \pm 2 \dots$ (integer)

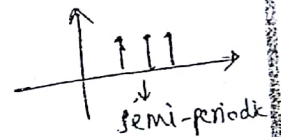
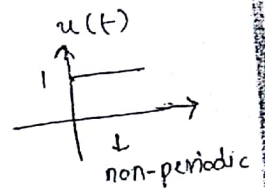
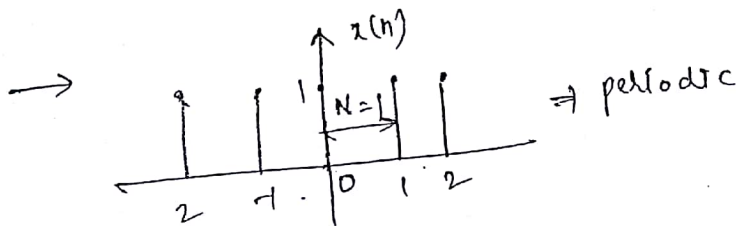
$T, N \rightarrow$ fundamental periods



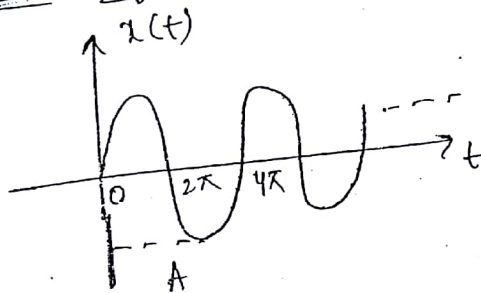
$$x(t) = 1$$



$\Rightarrow$ Non-periodic or aperiodic



Semiperiodic signals



Find the whether signals are periodic or non-periodic. If periodic indicate its fundamental period

1. $x(t) = 2\cos 20\pi t$

$$\omega_0 = 20\pi$$

$$\frac{2\pi}{T} = 20\pi$$

$$T = 0.1$$

$$x(t) = x(t \pm kT)$$

Let $k=1$

$$x(t) = x(t+T)$$

$$x(t+0.1) = 2\cos 20\pi(t+0.1)$$

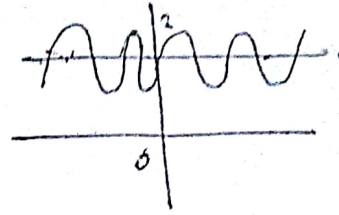
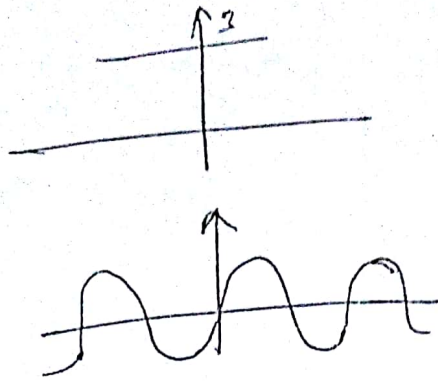
$$= 2[\cos 20\pi t \cos 2\pi - \sin 20\pi t \sin 2\pi]$$

$$= 2\cos 20\pi t = x(t)$$

Signal is periodic with period 0.1

$$(2) x(t) = 3 + \sin 2t$$

key-

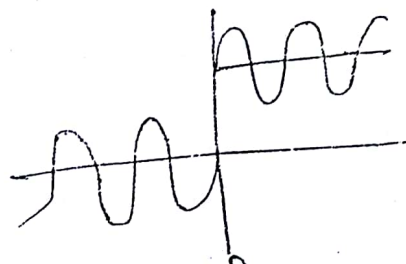
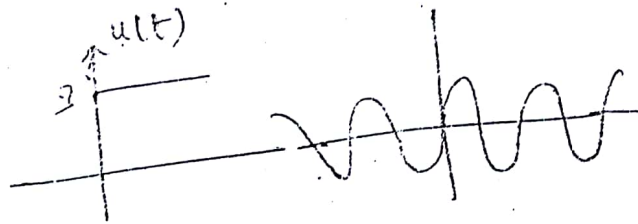


$$\omega = 2$$

$$\frac{2\pi}{T} = 2$$

$$T = \pi \text{ s}$$

$$(3) x(t) = 3u(t) + \sin 2t$$



⇒ non-periodic

$$(4) x(t) = e^{-j\left(\frac{3\pi}{5}t + \frac{\pi}{2}\right)}$$

$$= Ae^{j(\omega_0 t + \phi)}$$

⇒ complex exponential signal

$$\omega_0 = \frac{3\pi}{5}$$

$$\frac{2\pi}{T} = \frac{3\pi}{5}$$

$$T = \frac{10}{3} = 3.33 \text{ s}$$

$$\frac{3\pi}{5} \cdot \frac{10}{3} = 2\pi$$

$$\begin{aligned}
 x(t) &= x\left(t + \frac{10}{2}\right) \\
 &= e^{j\left[\frac{2\pi}{5}\left(t + \frac{10}{2}\right) + \frac{\pi}{2}\right]} \\
 &= e^{j\left[\frac{2\pi}{5}t + \frac{\pi}{2}\right]} \cdot e^{j\frac{2\pi}{5} \cdot \frac{10}{2}} \quad \left(\Rightarrow e^{-j2\pi}\right) \\
 &= e^{j\left(\frac{2\pi}{5}t + \frac{\pi}{2}\right)} \\
 \omega_0 &= \frac{2\pi}{5} \\
 \omega_0 &= m \left[\frac{2\pi}{N}\right] \\
 m &\rightarrow \text{any integer} \\
 N &\rightarrow \text{fundamental period.}
 \end{aligned}$$

$$\begin{aligned}
 * (5) \quad x(n) &= e^{j\left(\frac{2\pi}{5}n + \frac{\pi}{2}\right)} \\
 \omega_0 &= \frac{2\pi}{5} \\
 \frac{2\pi}{5} &= m \left[\frac{2\pi}{N}\right] \\
 \frac{2\pi}{5} &= 3 \left[\frac{2\pi}{10}\right] \\
 \boxed{N=10}
 \end{aligned}$$

is written in the form of

NOTE: Any discrete frequency $\omega_0 = m \left[\frac{2\pi}{N}\right]$ form

$$\begin{aligned}
 (6) \quad x(n) &= 2 \cos(0.04\pi n) \\
 0.04\pi &= 0.04 \left[\frac{2\pi}{2}\right] \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 0.04\pi &= 4 \left[\frac{2\pi}{200}\right] \quad \checkmark \\
 \frac{4\pi}{100} &= 4 \left[\frac{2\pi}{200}\right]
 \end{aligned}$$

$$\frac{4\pi}{100} = 1 \cdot \left[\frac{2\pi}{50}\right]$$

$$\boxed{N=50}$$

Fundamental period is always a $\checkmark$

$$(7) x(t) = e^{j(3/5)t}$$

$$\omega_0 = 3/5$$

$$\frac{2\pi}{T} = 3/5$$

$$T = \frac{10\pi}{3}$$

$$(8) x(n) = e^{j(3/5)n}$$

$$\omega_0 = 3/5$$

$$= \frac{2\pi}{16}$$

Signal is non-periodic

(1)

NOTE 1

[Any signal which cannot be expressed as $m(\frac{2\pi}{N}) \neq \omega_0$ it is non-periodic signal.]

Any discrete frequency cannot be expressed as $m(\frac{2\pi}{N})$ form then the signal is non-periodic

~~Ex~~ $x(t) \neq \cos t + \cos 2t$

Procedure for finding the periodicity of multiple signals:

(1) Find the individual fundamental periods and denote them as $T_1, T_2, T_3, \dots, T_n$ (CTS), $N_1, N_2, N_3, \dots, N_n$ (DTS),

(2)

(2) Obtain the ratios $\frac{T_1}{T_2}, \frac{T_1}{T_3}, \dots, \frac{T_1}{T_n}$; $\frac{N_1}{N_2}, \frac{N_1}{N_3}, \dots, \frac{N_1}{N_n}$

(3) $(T_1, T_2, T_3, \dots) \rightarrow$ Non-periodic if any one of the above ratios is an irrational number the signal is non-periodic and non-periodic signals has no fundamental periods.

(4) If all the above ratios are rational number the signal is periodic and its fundamental period is obtained as $T = \lambda T_1$; $N = \lambda N_1$,

$\lambda = \text{LCM of simplified denominators of above ratios.}$

$$(1) \quad x(t) = 1 + \cos(25\pi t) + \sin(40\pi t) + \cos(15\pi t)$$

$$\omega_1 = 25\pi,$$

$$\omega_2 = 40\pi,$$

$$\omega_3 = 15\pi$$

$$\frac{2\pi}{T_1} = 25\pi$$

$$\frac{2\pi}{T_2} = 40\pi$$

$$\frac{2\pi}{T_3} = 15\pi$$

$$(1) \quad T_1 = 2/25,$$

$$T_2 = 1/20,$$

$$T_3 = 2/15$$

$$(2) \quad \frac{T_1}{T_2} = \frac{2}{25} \times \frac{20}{1}$$

$$= \frac{8}{5}$$

$$\frac{T_1}{T_3} = \frac{2}{25} \times \frac{15}{2}$$

$$= 3/5$$

$$(3) \quad T = 2 \times T_1$$

$$x = 2 \text{ cm } \angle 5.5^\circ$$

$$= 5$$

$$T = 2 \times \frac{2}{25} = \frac{2}{5} = 0.4 \text{ sec}$$

$$(2) \quad x(t) = \cos 2t + \cos 3t$$

$$\omega_1 = 2,$$

$$\omega_2 = 3$$

$$\frac{2\pi}{T_1} = 2$$

$$\frac{2\pi}{T_2} = 3$$

$$(1) \quad T_1 = \pi,$$

$$T_2 = \frac{2\pi}{3}$$

$$(2) \quad \frac{T_1}{T_2} = \pi \times \frac{3}{2\pi} = 3/2$$

$$(3) \quad T = 1 \times T_1$$

$$= 2 \times \pi$$

$$= 2\pi \text{ sec}$$

3. Fourier Series Expansion of a periodic signal is given as
 $x(t) = \cos(1.2\pi t) + \cos(2\pi t) + \cos(2.8\pi t)$ then the fundamental frequency of signal is?

Soln

$$x(t) = \cos(1.2\pi t) + \cos(2\pi t) + \cos(2.8\pi t)$$

$$\omega_1 = 1.2\pi, \quad \omega_2 = 2\pi, \quad \omega_3 = 2.8\pi$$

$$\frac{2\pi}{T_1} = 1.2\pi, \quad \frac{2\pi}{T_2} = 2\pi, \quad \frac{2\pi}{T_3} = 2.8\pi$$

$$(1) \quad T_1 = \frac{2}{1.2}, \quad T_2 = 1, \quad T_3 = \frac{2}{2.8}$$

$$(2) \quad \frac{T_1}{T_2} = \frac{2}{1.2}, \quad \frac{T_1}{T_3} = \frac{2}{1.2} \times \frac{2.8}{2} = \frac{7}{3}$$

$$= \frac{20}{12} = \frac{5}{3}$$

$$(3) \quad T = 2 \times T_1$$

$$= 3 \times \frac{5}{3}$$

$$= 5$$

$$\boxed{f = \frac{1}{5} = 0.2 \text{ Hz}}$$

4. The frequency of the 5th Harmonic for the above problem

$$f = 0.2 \text{ Hz}$$

$$5\text{th Harmonic} = 5f_0 = \underline{\underline{1 \text{ Hz}}}$$

$$5. \quad x(n) = e^{j\frac{3\pi n}{4}} + e^{j\frac{4\pi n}{5}}$$

$$\omega_1 = \frac{3\pi}{4} = 2\left(\frac{2\pi}{8}\right)$$

$$\omega_2 = \frac{4\pi}{5} = 2\left(\frac{2\pi}{5}\right)$$

$$N_2 = 5$$

$$(1) \quad N_1 = 8$$

$$(2) \quad \frac{N_1}{N_2} = 8/5$$

$$(3) \quad N = 2 \times N_1$$

$$= 5 \times 8 = 40 \text{ sec}$$

Deterministic and Random signals?

Eg: $x(n) = 2n+3$

→ A signal whose o/p can be estimated at any instant of time are known as deterministic signal. These signals are represented in some mathematical forms.

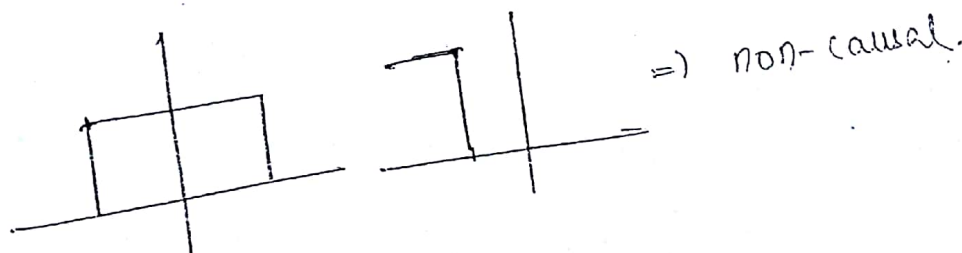
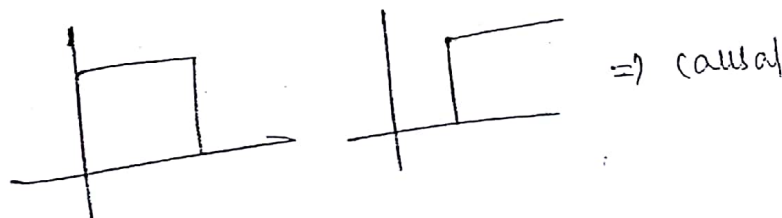
Random signal: A signal whose o/p cannot be estimated at any instant of time is known as random signal.

Eg: noise, speech signals etc.

Note: More the randomness more is the signal content.

→ Causal and non-causal signals:

Causal signal: Any signal whose time period starts after a pre-determined time $t > 0$ for continuous time signals
 $n \geq 0$ for discrete time signals.



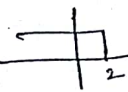
Note:

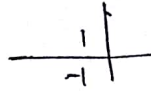
- (1) All the right side signals are known as causal signals
- (2) 2-sided and left sided are non-causal signals
- (3) Left sided signals are more specifically known as anti-causal signals

Which of the following signals are causal, non-causal and anti-causal

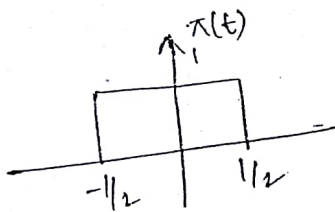
(a) $x(t) = t^2 \delta(2t-3) \Rightarrow t = 3/2 \Rightarrow$ causal

(b) $x(t) = e^{-\alpha t} [\delta(t+2) - \delta(t-2)]$, $-2 < t < 2 \Rightarrow$ non-causal

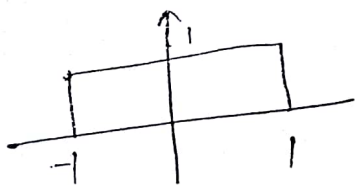
(c) $h(n) = e^{2n} 4(2-n)$, $-\infty \leq n \leq 2$,  $\Rightarrow$ non-causal

(d) $x(n) = \frac{n}{2} \delta(-n-1)$, non-causal or anti-causal $-\infty < n < -1$ 

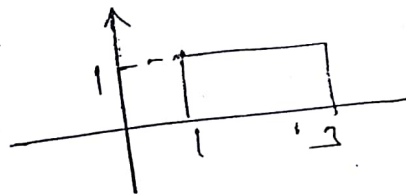
(e) $h(t) = \pi \left(\frac{t-2}{2} \right) = \pi \left(\frac{t}{2} - 1 \right) = \pi \left(\frac{1}{2}(t-2) \right)$
~~not causal~~



$\pi(t/2)$



$\pi \left(\frac{1}{2}(t-2) \right)$

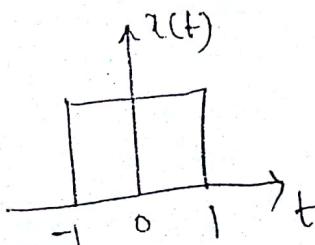


Causal, $1 < t < 3$

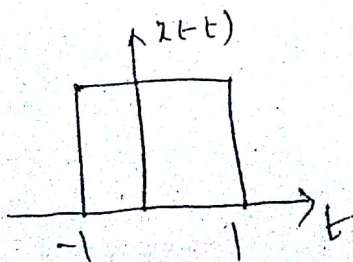
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Even and odd signal classification

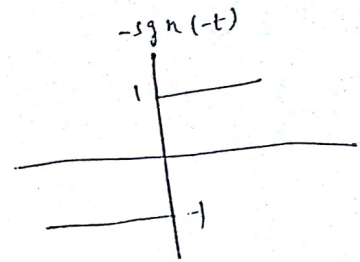
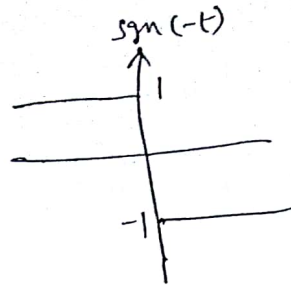
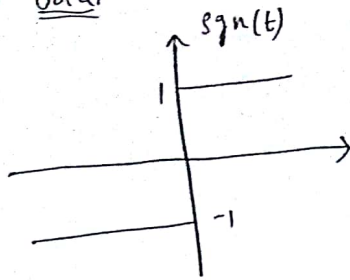
Even:



$x(t) = x(-t)$
 $x(n) = x(-n)$ } even signals (or) symmetric signals

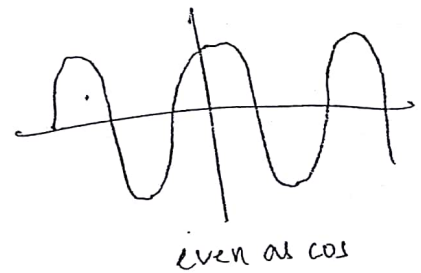
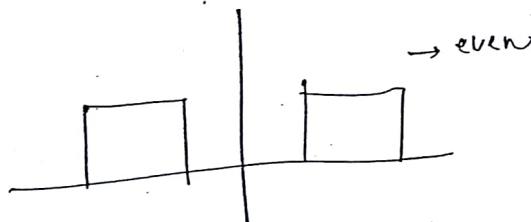


odd



$$\left. \begin{aligned} x(t) &= -x(-t) \\ x(n) &= -x(-n) \end{aligned} \right\} \text{odd (or) antisymmetric signal.}$$

*



$$x(t) = t^2 \cos 2t \text{ (even)}$$

$$x(t) = \sin 2t + \sin 3t \text{ (odd)}$$

$$x(t) = \cos 2t + \sin 2t$$

(neither odd nor even)

NOTE: Any signal is always a combination of even or odd parts. If the signal is an even then it has zero odd parts & vice versa

$$x(t) = x_e(t) + x_o(t) \text{ — (1)}$$

$$x(-t) = x_e(-t) + x_o(-t)$$

$$= x_e(t) - x_o(t) \text{ — (2)}$$

Adding ① & ②

$$x_e(t) = \frac{x(t) + x(-t)}{2} \quad (\text{even})$$

$$x_o(t) = \frac{x(t) - x(-t)}{2} \quad (\text{odd})$$

* find even and odd parts of following signals.

$$x(n) = \{-3, 0, 5, 2, 1\}$$

$$x(-n) = \{1, 2, 5, 0, -3\}$$

$$x_e(n) = \{-1, 1, 5, 1, -1\} \quad \text{even}$$

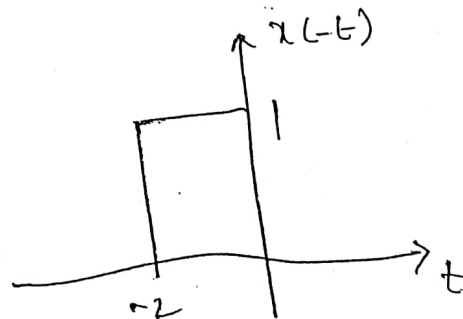
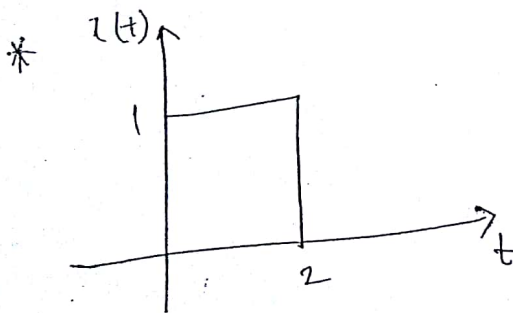
$$x_o(n) = \{-2, -1, 0, 1, 2\} \quad \text{odd}$$

* $x(t) = (1-t+t^2) \cos 3t$

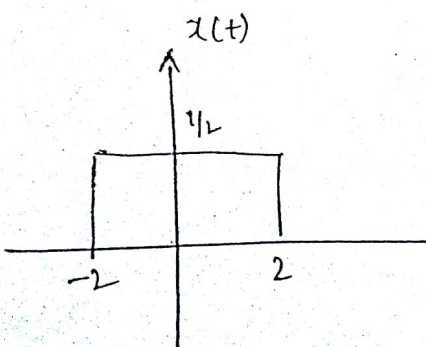
$$x(-t) = (1+t+t^2) \cos 3t$$

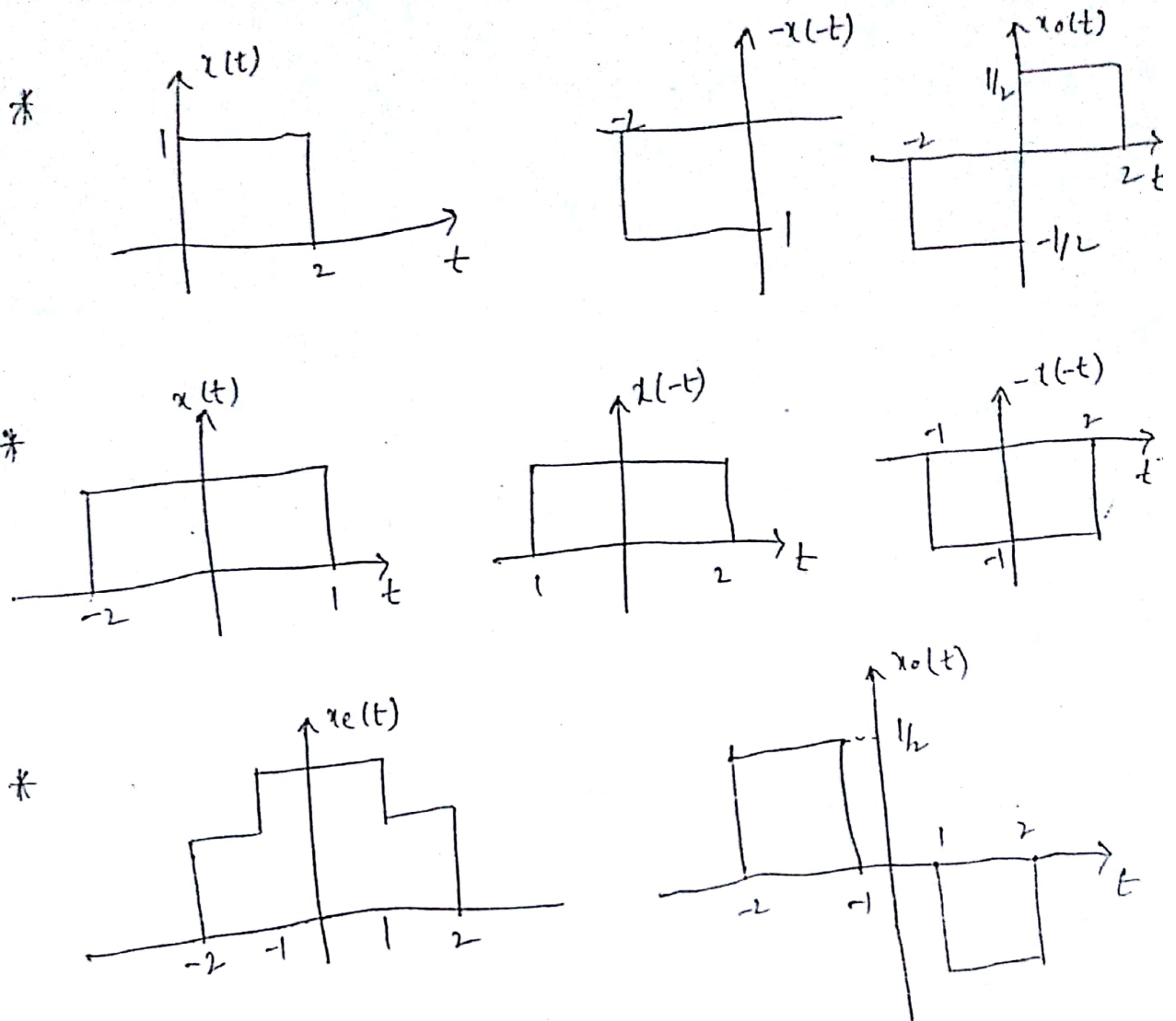
$$x_e(t) = (1+t^2) \cos 3t$$

$$x_o(t) = -t \cos 3t$$



$$x_e(t) = \frac{x(t) + (x(-t))}{2}$$





Energy and power signals

Energy signal: A signal is said to be an energy signal if it radiates finite energy given as $0 < E < \infty$. The energy of signal is calculated as

$$E_{x(t)} = \lim_{T \rightarrow \infty} \int_{-T}^T |x(t)|^2 dt$$

$$E_{x(n)} = \lim_{N \rightarrow \infty} \sum_{n=-N}^N |x(n)|^2$$

Power: A signal is said to be a power signal if it radiates finite average power.

$$0 < P < \infty$$

Power of the signal is calculated as

$$P_x(n) = \lim_{N \rightarrow \infty} \frac{1}{(2N+1)} \sum_{n=-N}^N |x(n)|^2$$

$$P_x(t) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt$$

Note 1 If the signal is a power signal then it radiates infinite energy.

→ If the signal is an energy signal then it possess zero avg. power

→ All the bounded signals possess finite energy and periodic signals possess finite avg. power.

→ If atleast any of the above conditions are not possessed then they are neither energy nor power signals.

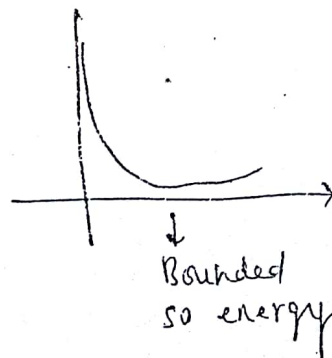
→ All periodic signals are power signals, bounded signals are energy signals.

Problems

find whether energy / power and also find their energy or power.

$$\begin{aligned} 1. \quad E &= \lim_{T \rightarrow \infty} \int_{-T}^T |e^{-2t} u(t)|^2 dt \\ &= \lim_{T \rightarrow \infty} \int_0^T |e^{-4t}| dt \end{aligned}$$

$$= \lim_{T \rightarrow \infty} \left| \frac{e^{-4t}}{-4} \right|_0^T = \lim_{T \rightarrow \infty} \frac{e^{-4T} - 1}{-4} = \frac{1}{4}$$



$$\begin{aligned}
 P_x(t) &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt \\
 &= \lim_{T \rightarrow \infty} \frac{1}{2T} \left(\frac{e^{-4t} - 1}{-4} \right) \\
 &= 0
 \end{aligned}$$

2. $x(t) = 10e^{-j(2\pi/3)t + \pi/2}$ → complex exponential signal

$$= \lim_{T \rightarrow \infty} \int_{-T}^T (10e^{-j(2\pi/3)t + \pi/2})^2 dt$$

$$= \lim_{T \rightarrow \infty} \int_{-T}^T 100e^{-j(4\pi/3)t + \pi} dt$$

$$= \lim_{T \rightarrow \infty} \int_{-T}^T 100(1) dt = \lim_{T \rightarrow \infty} 100(2T) = \infty$$

$$P_{av} = \lim_{T \rightarrow \infty} \frac{1}{2T} (200T) = 100$$

3. $x(t) = t^4 u(t)$

$$E = \lim_{T \rightarrow \infty} \int_{-T}^T |t^4 u(t)|^2 dt$$

$$= \lim_{T \rightarrow \infty} \int_0^T t^4 dt = \lim_{T \rightarrow \infty} \frac{t^5}{5} \bigg|_0^T = \lim_{T \rightarrow \infty} \frac{T^5}{5} = \infty$$

$$P_{av} = \lim_{T \rightarrow \infty} \frac{1}{2T} \cdot \frac{T^5}{5}$$

$$= \frac{1}{10} \lim_{T \rightarrow \infty} T^4$$

$$= \infty$$

$$\sum_{r=0}^{\infty} a^r = \frac{1}{1-a}; |a| < 1$$

neither energy nor power signals.

$$4. \quad x(n) = \left(\frac{1}{2}\right)^n u(n)$$

$$E = \lim_{T \rightarrow \infty} \sum_{-N}^N \left| \left(\frac{1}{2}\right)^n u(n) \right|^2$$

$$= \lim_{T \rightarrow \infty} \sum_0^{\infty} \left| \frac{1}{2} \right|^{2n}$$

$$= \lim_{T \rightarrow \infty} \left[\left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \dots \right]$$

$$= \lim_{T \rightarrow \infty} (1 - \frac{1}{2})^{-1}$$

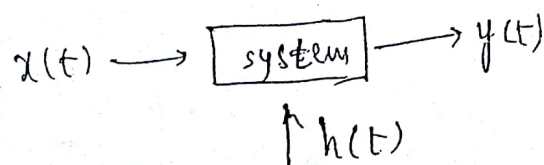
$$= \frac{1}{1 - (\frac{1}{2})} = 2$$

$$5. \quad x(n) = 2^n [u(n) - u(n-3)] \quad 0 < n < \infty$$

$$= \lim_{N \rightarrow \infty} \sum_{N=0}^2 [2^n [u(n) - u(n-3)]]$$

$$= \lim_{N \rightarrow \infty} \sum_{N=0}^2 2^n = \frac{1 - 2^{3+1}}{1 - 2}$$

System: A system is a physical parameter which converts input response into a valid output response as shown in figure below



$$y(t) = T(x(t))$$

convolution: The process of obtaining system response by continuous mixing of input response with impulse response.

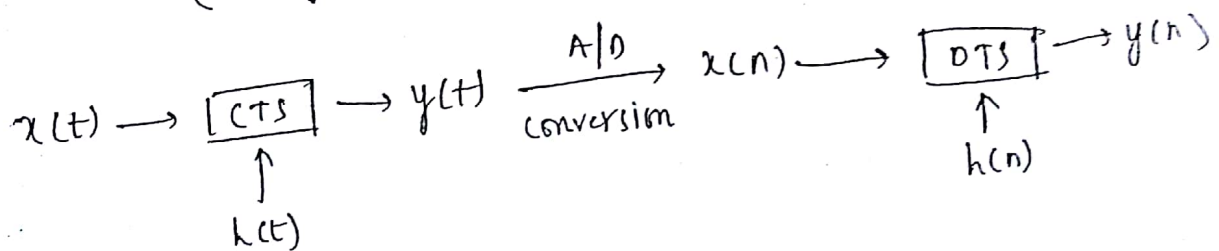
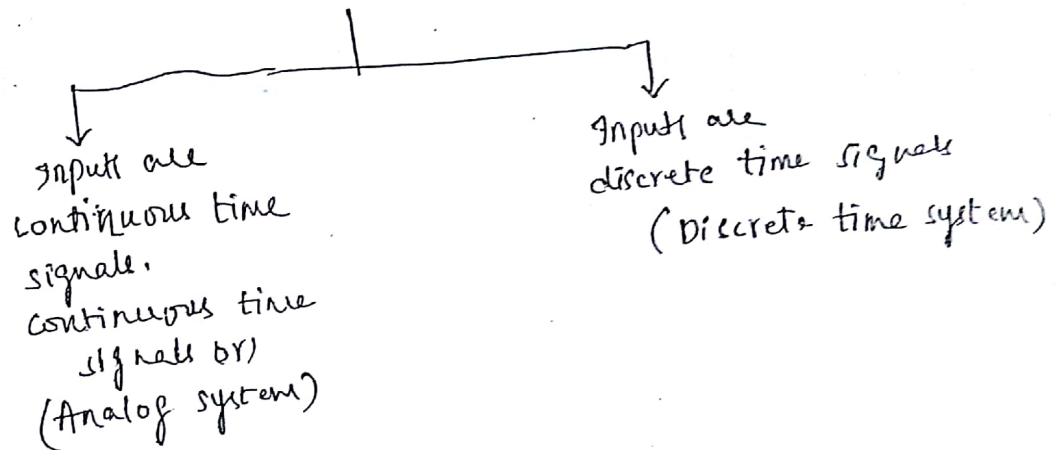
$$y(t) = x(t) * h(t)$$

$x(t) \rightarrow$ i/p response

$h(t) \rightarrow$ impulse response

$y(t) \rightarrow$ o/p response

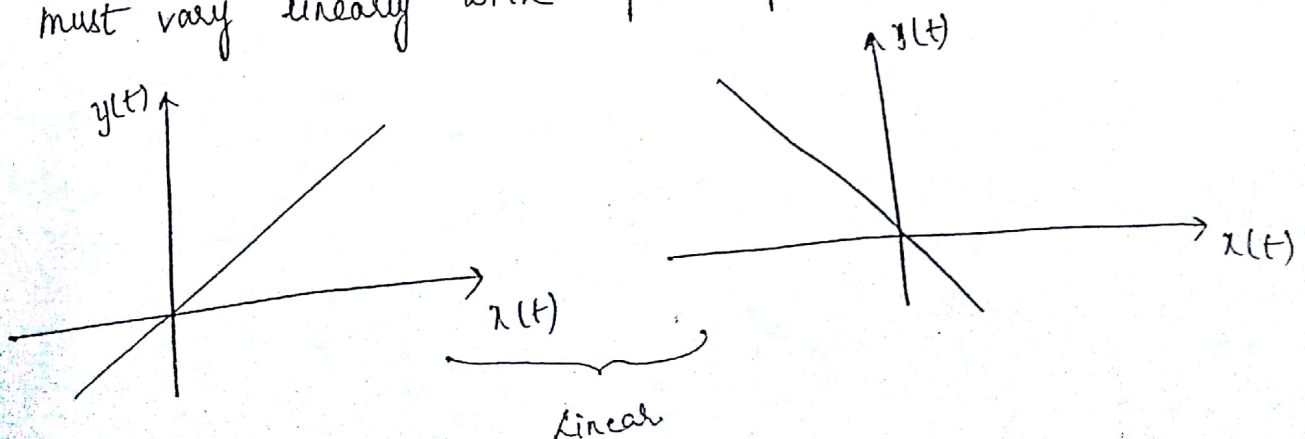
* Basic classification of system:

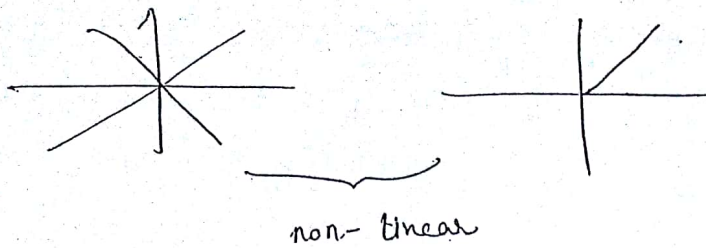


Complete classification of system:

1. Linear and Non-Linear system classification:

For the system to be linear the output response must vary linearly with input response as shown below.





They must satisfy two conditions.
Homogeneity & superposition (additivity)

Homogeneous:

$$x(t) \rightarrow y(t) \text{ and } ax(t) \rightarrow ay(t)$$

Superposition or additivity:

$$x_1(t) \rightarrow y_1(t) \text{ and } y_2(t)$$

$$x_2(t) \rightarrow y_2(t) \text{ then}$$

$$x_1(t) + x_2(t) \rightarrow y_1(t) + y_2(t).$$

on combining both the conditions we get

$$ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t).$$

So, if above condition is satisfied then system is linear

* Find whether the system is linear or non-linear

$$y(n) = e^{x(n)}$$

$$y_1(n) = e^{x_1(n)}$$

$$y_2(n) = e^{x_2(n)}$$

$$y_3(n) = e^{x_3(n)}$$

$$y_3(n) = ay_1(n) + by_2(n)$$

$$x_3(n) = ax_1(n) + bx_2(n)$$

$$ay_1(n) + by_2(n) = e^{[ax_1(n) + bx_2(n)]} \text{ --- (1)}$$

$$\text{But } ay_1(n) + by_2(n) = ae^{x_1(n)} + be^{x_2(n)} \text{ --- (2)}$$

At (1) + (2)

It is a non-linear system

$$* \quad y(t) = 2x(t) + 3$$

$$y_1(t) = 2x_1(t) + 3$$

$$y_2(t) = 2x_2(t) + 3$$

$$y_3(t) = 2x_3(t) + 3$$

$$y_3(t) = ay_1(t) + by_2(t)$$

$$= 2ax_1(t) + 3a + 2bx_2(t) + 3b \text{ --- (1)}$$

$$y_3(t) = 2ax_1(t) + 2bx_2(t) + 3 \text{ --- (2)}$$

(1) + (2) non-linear system

$$y(t) = t^2 x(t)$$

$$y_1(t) = t^2 x_1(t)$$

$$y_2(t) = t^2 x_2(t)$$

$$y_3(t) = t^2 x_3(t) \Rightarrow y_3 = y_1 + y_2 = t^2 [x_1(t) + x_2(t)] \text{ --- (1)}$$

$$ay_1 + by_2 = at^2 x_1(t) + bt^2 x_2(t) \text{ --- (2)}$$

(1) = (2) linear

* $\rightarrow$ If the system response is in the form of trigonometric exponential, logarithmic, inverse, squares, square roots, cubes, cube roots, power more than one, nth power more than one, $y = mx + c$ modulus are all non-linear responses.

Time invariant systems and Time variant systems

$\Rightarrow$ The system whose output response does not change with time are known as Time invariant system

⇒ The system whose output response ~~whose~~ changes with time

$$\text{Eg)} \quad y(t) = x(t) \rightarrow x(t-t_0)$$

$$y(t-t_0) = 2(t-t_0)$$

T. I

$$y(t) = t x(t)$$

$$t \rightarrow [t - t_0]$$

$$y(t-t_0) = (t-t_0) \lambda (t-t_0)$$

T.V

- $\Rightarrow$ (1) LTI $\rightarrow$ Linear + T-I
 (2) LTV $\rightarrow$ Linear + T-V
 (3) Non LTV $\rightarrow$ Non-Linear + T-V
 (4) Non-LTI $\rightarrow$ Non-Linear + Time invariant

* Static & Dynamic systems

Static: The systems whose o/p response depends on only on the present values of i/p response. The systems are static or memoryless systems.

Eg: $y(n) = x(n)$

$$n=0; \quad y(0)=x(0)$$

$$n=1; \quad y(1) = x(1)$$

$-3, -2, -1$ 0 $1, 2, 3$
 $\underbrace{\hspace{1.5cm}}$ $\downarrow$ $\underbrace{\hspace{1.5cm}}$
 past present future

Dynamic: The system whose o/p depends on either past or future or both values of i/p response. The systems are dynamic or with memory systems.

Eg: $y(n) = x(n) + x(n+1)$
 $n=0 \Rightarrow y(0) = x(0) + x(1)$

* $\frac{dy(t)}{dt} + 5y(t) = 3tx(t)$ [Dynamic]

NOTE: $\frac{d}{dt}$, $\int$, D , Σ , these all are dynamic elements.

* Causal and Non-causal system

Causal: Any system whose o/p depends on either present or past values of the input response. Eg: $y(n) = x(n) + x(n-1)$
 $y(0) = x(0) + x(-1)$

Non-causal: Any system whose o/p depends on atleast one value on the future input.

Eg: $y(n) = x(n+1)$
 for $n=0 \Rightarrow y(0) = x(1)$

→ Classify the following systems for LTI, LTV, NLTI, NLTV, Static, Dynamic, causal and non-causal.

1. $\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 2y(t) = t^2x(t) \Rightarrow$ LTV, Dynamic, causal

2. $y(n) = x(2n) \Rightarrow$ LTV, Dynamic, non-causal

3. $y(t) = \dot{x}(t) + x(t) \Rightarrow$ NLTV, Dynamic, non-causal

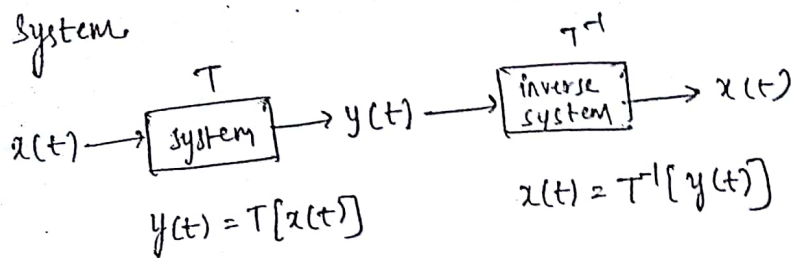
4. $y(t) = 2x(t) \cos t \Rightarrow$ NLTV, Static, causal

5. $y(n)y(n-1) = x(n)$; NLTI, Dynamic, Non-causal.
 $\downarrow$
 $x(n-n_0)$

$$y(n-n_0)y(n-n_0-1) = x(n-n_0)$$

* Reversible and Irreversible or Inverse and Non-Inverse system

Reversible A system is said to be reversible if the o/p response
 reverse produces the same i/p response given to the
 system.



Eg) $y(t) = \frac{1}{2}x(t)$

$$= T[x(t)]$$

$$T = \frac{1}{2}, \quad T^{-1} = 2$$

$$x(t) = T^{-1}[y(t)]$$

$$= 2 \left[\frac{1}{2}x(t) \right]$$

$$= x(t)$$

$\therefore$ Reversible

$\Rightarrow$ The systems are said to be reversible only when the
 distinct i/p produces distinct o/p.

$$y(t) = \frac{d}{dt}x(t)$$

same o/p $\Rightarrow$ reversible

$\int \Rightarrow$ irreversible

$$y(t) = x^2(t)$$

$T \rightarrow \text{square}$

$$T^{-1} = \sqrt{\quad}$$

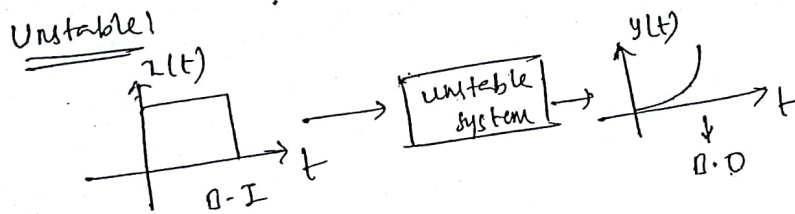
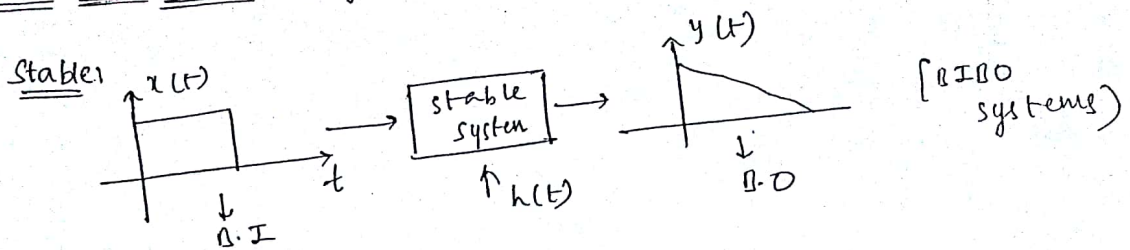
$$x(t) = T^{-1}[y(t)]$$

$$= \sqrt{x^2(t)}$$

$$= \pm x(t)$$

Irreversible

* Stable and Unstable system



$$\rightarrow \int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty \rightarrow \text{C.T.S}$$

$$\rightarrow \sum_{k=-\infty}^{\infty} |h(k)| < \infty \rightarrow \text{D.T.S}$$

* $y(n) = 2x(n) + 3x(n-1)$

If $x(n) = \delta(n)$; then $y(n) = h(n)$

$$h(n) = 2\delta(n) + 3\delta(n-1)$$

$$\sum_{k=-\infty}^{\infty} |h(k)| = 2+3 = 5$$

$\therefore$ stable system

The condition for the system response $h(t) = e^{-at}u(t)$ to be stable is?

$$h(t) = e^{-at}u(t)$$

$$\int_{-\infty}^{\infty} |h(t)| dt = \int_{-\infty}^{\infty} e^{-at}u(t) dt$$

$$= \left[\frac{e^{-at}}{-a} = -\frac{1}{a}e^{-at} \right]_{0}^{\infty} = \frac{1}{a}$$

$\therefore$ condition is $a > 0$

$a > 0$

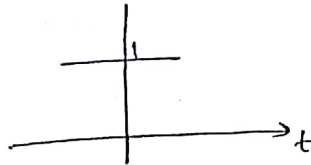
Bounded

$$y(t) = \int_{-\infty}^t \cos 3T x(T) \cdot dT$$

unstable system

$$h(t) = \int_{-\infty}^t \cos 3T \delta(t) \cdot dT$$

$$= 1$$



$$\int_{-\infty}^{\infty} 1 \cdot dT = \infty$$

$$\left. \frac{\sin 3T}{3} \right|_{-\infty}^T$$

$$-\frac{\sin 3(-\infty)}{3} + \frac{\sin 3T}{3}$$

High

→

→

→

→ for

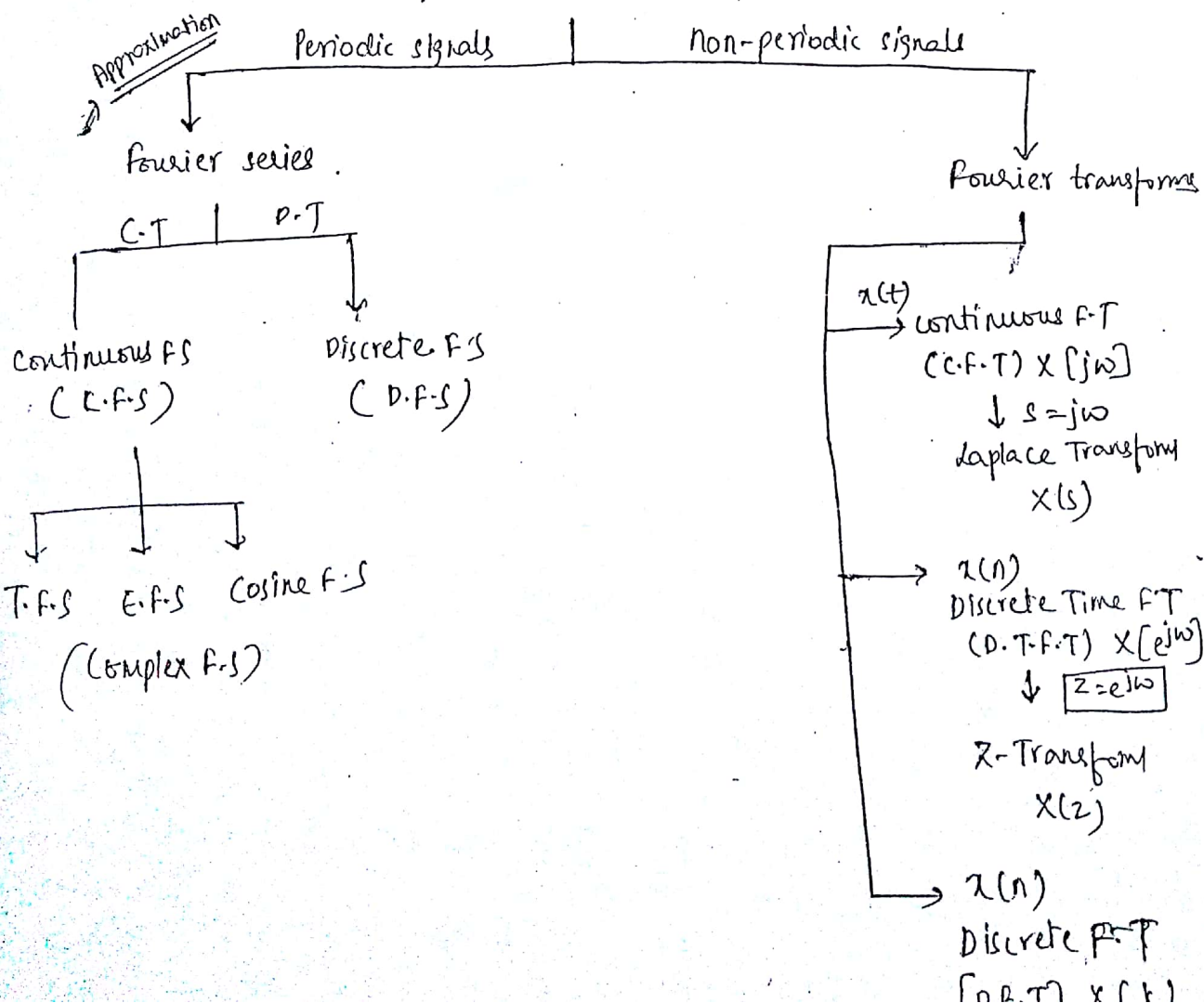
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Transformation

* Transformation: obtaining the freq. characteristics from a time domain signal

* Anal

→ Orl



Highlights of Transformation:

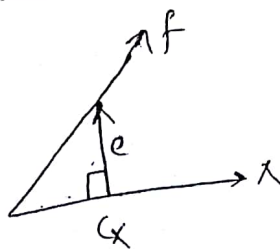
- The periodic signals are converted into discrete spectrum
- The non-periodic signals are converted into continuous spectrum
- Since, D.S.F cannot analyse the C.Spectrum So, the cont. spectrum in P.T.F.T is converted into discrete spectrum known as D.F.T.

→ Fourier Transforms are only for bounded signals and stable systems. So, in order to analyse both stable and unstable systems, bounded and unbounded signals Laplace and Z-Transforms are used.

* Analogy b/w vectors and signals

→ Orthogonality in vectors

Let f and x be two vectors and c_x be another vector which is a component of f along the x -direction. Our main aim is to make a minimum error approximation between the two vectors as shown in the fig below.



By addition of vectors $\Rightarrow f = c_x + e$

$$\boxed{e = f - c_x} \quad \text{--- (1)}$$

The vectors f and x are $\perp$ if $\theta = 90^\circ$

$$f \cdot x = |f||x|\cos\theta$$

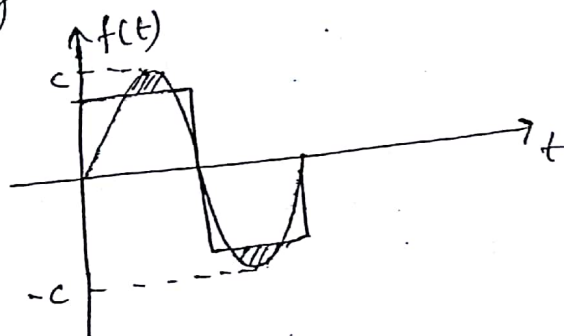
$$\boxed{f \cdot x = 0} \quad \text{--- (2)}$$

So, if the dot product b/w the vectors is zero.

The vectors are known as orthogonal vector $\therefore$ the error b/w them is $\boxed{e=0}$

Orthogonality in signals

Let $f(t)$ and $x(t)$ be two signals over an interval $t_1 < t < t_2$. Let $cx(t)$ is a component of $f(t)$ along $x(t)$. Our main aim is to find a minimum error existing b/w the signals.



$$e = f(t) - cx(t) \quad \text{--- (1)}$$

$$E_e = \int_{t_1}^{t_2} [f(t) - cx(t)]^2 dt$$

$$= \int_{t_1}^{t_2} f^2(t) dt - 2c \int_{t_1}^{t_2} f(t) \cdot x(t) dt + c^2 \int_{t_1}^{t_2} x^2(t) dt$$

$$\frac{dE_e}{dc} = 0; \quad 0 = 0 - 2 \int_{t_1}^{t_2} f(t) \cdot x(t) \cdot dt + 2c \int_{t_1}^{t_2} x^2(t) dt = 0$$

$$c = \frac{\int_{t_1}^{t_2} f(t) \cdot x(t) \cdot dt}{\int_{t_1}^{t_2} x^2(t) \cdot dt}$$

Proble

Let

a

Soln

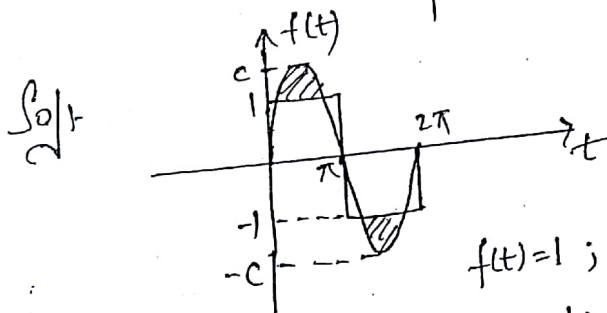
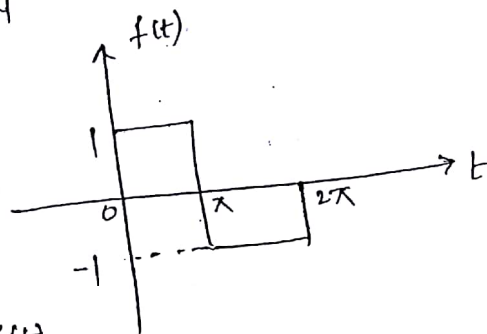
→ If the error const. $c=0$ then the two signals are said to be orthogonal to each other.

If $c=0$;

$$\int_{-t_1}^{t_2} f(t) \cdot z(t) dt = 0$$

Problem:

Let $f(t)$ be a signal (rect) as defined below, approximate another signal $c \sin t$ such that a minimum error exist b/w signal



$$f(t) = 1; \quad 0 < t < \pi$$

$$f(t) = -1; \quad \pi < t < 2\pi$$

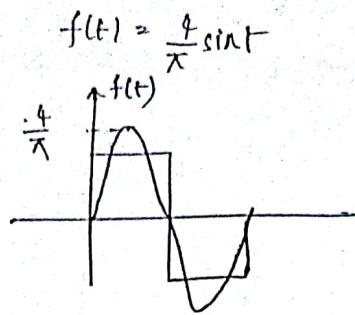
$$c = \frac{\int_0^{\pi} 1 \cdot \sin t dt + \int_{\pi}^{2\pi} (-1) \sin t dt}{\int_0^{2\pi} \sin^2 t dt} = \frac{\cos t \Big|_0^{\pi} + \cos t \Big|_{\pi}^{2\pi}}{\int_0^{2\pi} \sin^2 t dt} = \frac{-[1-1] + [1-1]}{\int_0^{2\pi} \sin^2 t dt} = 4$$

$$\int_0^{2\pi} \sin^2 t dt$$

$$= \int_0^{2\pi} \frac{1 - \cos 2t}{2} dt = \frac{1}{2} (2\pi) - \frac{1}{2} \left[\frac{\sin 2t}{2} \right]_0^{2\pi}$$

$$= \pi - \frac{1}{4} [0]$$

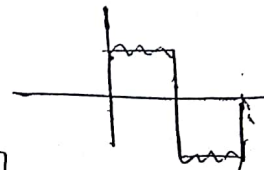
$$= 4/\pi$$



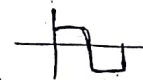
$$f(t) = \frac{4}{\pi} \left[\sin t + \frac{1}{3} \sin 3t \right]$$



$$f(t) = \frac{4}{\pi} \left[\frac{1}{1} \sin t + \frac{1}{3} \sin 3t + \frac{1}{5} \sin 5t \right]$$



$$f(t) = \frac{4}{\pi} \left[\frac{1}{1} \sin t + \frac{1}{3} \sin 3t + \frac{1}{5} \sin 5t + \frac{1}{7} \sin 7t + \dots \right]$$



→ Fourier series expansion
complete orthogonal set of
functions

NOTE Any periodic signal can be expanded as sum of its orthogonal functions. Such expansion of a periodic signal as sum of its orthogonal functions, is known as the Fourier series expansion.

**
* I

Trigonometric Fourier Series [T.F.S]:

$$f(t) = a_0 + a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t + \dots + a_n \cos n\omega_0 t + b_1 \sin \omega_0 t + b_2 \sin 2\omega_0 t + \dots + b_n \sin n\omega_0 t$$

$$a_0 = \text{average value} = \frac{1}{T} \int_0^T f(t) \cdot dt = \left[\frac{\text{area}}{T} \right]$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos n\omega_0 t + b_n \sin n\omega_0 t]$$

$$a_n = \frac{2}{T} \int_0^T f(t) \cos n\omega_0 t \cdot dt$$

$$b_n = \frac{2}{T} \int_0^T f(t) \sin n\omega_0 t \cdot dt$$

→ a_0, a_n, b_n → T.F.S Co-efficients.

→ $\omega_0, 2\omega_0, 3\omega_0, \dots, n\omega_0$ → Fundamental or Harmonic or orthogonal frequencies.

Cosine F.S

$$f(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos [n\omega_0 t - \phi_n]$$

$$A_0 = a_0; \quad A_n = \sqrt{a_n^2 + b_n^2}; \quad \phi_n = \tan^{-1} \left(\frac{b_n}{a_n} \right)$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \cos \left[n\omega_0 t - \tan^{-1} \left(\frac{b_n}{a_n} \right) \right]$$

$$\cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

** Exponential F.S

$$f(t) = a_0 + a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t + \dots + a_n \cos n\omega_0 t + b_1 \sin \omega_0 t + b_2 \sin 2\omega_0 t + \dots + b_n \sin n\omega_0 t$$

$$C_0 = a_0 \approx \text{avg. value} = \frac{1}{T} \int_0^T f(t) \cdot dt = \left[\frac{\text{area}}{T} \right]$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos n\omega_0 t + b_n \sin n\omega_0 t]$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \left(\frac{e^{-jn\omega_0 t} + e^{jn\omega_0 t}}{2} \right) + \sum_{n=1}^{\infty} b_n \left(\frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j} \right)$$

$$= a_0 + \sum_{n=1}^{\infty} e^{jn\omega_0 t} \left(\frac{a_n}{2} + \frac{b_n}{2j} \right) + \sum_{n=1}^{\infty} e^{-jn\omega_0 t} \left(\frac{a_n}{2} - \frac{b_n}{2j} \right)$$

$$= a_0 + \sum_{n=1}^{\infty} \left(\frac{a_n - j b_n}{2} \right) e^{j n \omega_0 t} + \sum_{n=1}^{\infty} \left(\frac{a_n + j b_n}{2} \right) e^{-j n \omega_0 t}$$

$\downarrow$ $\downarrow$ $\downarrow$
 G_0 C_n C_{-n}

$$= G_0 + \sum_{n=1}^{\infty} C_n e^{j n \omega_0 t} + \sum_{n=1}^{\infty} C_n e^{-j n \omega_0 t}$$

$$= \sum_{n=-\infty}^{-1} C_n e^{j n \omega_0 t} + G_0 + \sum_{n=1}^{\infty} C_n e^{j n \omega_0 t}$$

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} C_n e^{j n \omega_0 t}} \rightarrow \text{E.F.S}$$

$C_0, C_n, C_{-n} \rightarrow$ EFS coefficients

$$C_n = \frac{1}{T} \int_0^T f(t) e^{-j n \omega_0 t} dt$$

* Relation b/w T.F.S and E.F.S coefficients:

$$C_0 = a_0$$

$$C_n = \frac{a_n - j b_n}{2} \quad \text{--- (1)}$$

$$C_{-n} = \frac{a_n + j b_n}{2} \quad \text{--- (2)}$$

conjugate pairs

$$\textcircled{1} + \textcircled{2} \Rightarrow a_n = C_n + C_{-n} \quad \text{--- (3)}$$

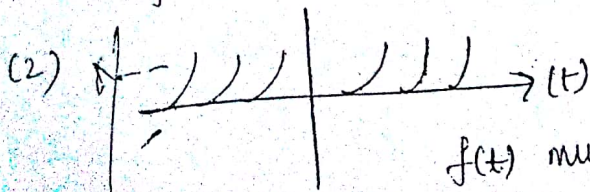
$$\textcircled{1} - \textcircled{2} \Rightarrow b_n = j(C_n - C_{-n}) \quad \text{--- (4)}$$

$\rightarrow$ Convergence of Fourier series or Conditions for obtaining P.S

[Dirichlet's Conditions]

$\rightarrow$ Necessary conditions

(1) $\rightarrow f(t)$ must be periodic



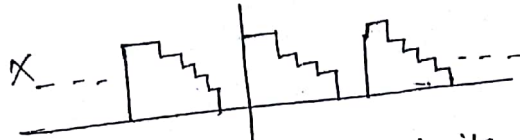
$f(t)$ must be absolutely integrable.
i.e. $f(t)$ must be bounded

Sufficient conditions:

1. $f(t)$ must have the finite no. of discontinuity in the signal



2.



$f(t)$ must have the finite no. of maxima and minima values in the signal.

* Spectrum: The graphical representation of a signal in its frequency domain is known as spectrum i.e. frequency response

$$\text{freq. response} = \text{Magnitude response} + \text{Phase response}$$

$$\downarrow \qquad \qquad \qquad \downarrow$$

$$\sqrt{Re^2 + Im^2} \qquad \qquad \tan^{-1} \left[\frac{Im}{Re} \right]$$

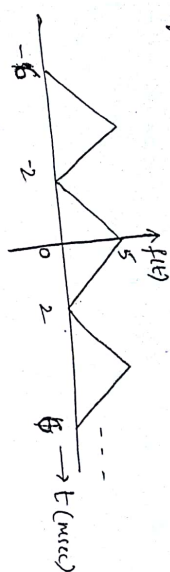
**NOTE The spectrum of fourier series is always discrete frequency response

→ In T.F.S, the frequency response is a causal response where it has only magnitude response and phase response is '0'.

→ In E.F.S, the spectrum is a two-sided F.R where its magnitude is symmetrical response and phase is anti-symmetrical response

Problems:

(1) find the frequency of 5th harmonic in the expansion of following periodic signal and also calculate its avg value



$$T = [2 - (-2)] = 4 \text{ msec}$$

$$f_0 = \frac{1}{T} = \frac{1}{4 \times 10^{-3}} = 250 \text{ Hz}$$

$$5f_0 = 5 \times 250 = 1250 \text{ Hz}$$

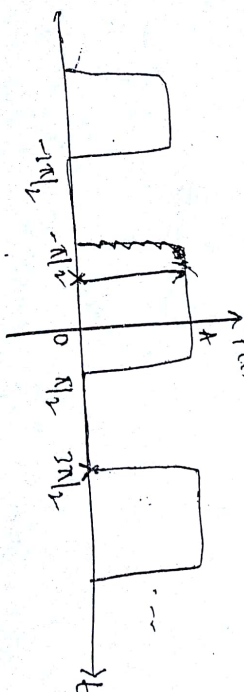
$$\text{Avg value} = \frac{1}{T} \int_0^T f(t) \cdot dt$$

$$f(t) = 2.5(t+2); -2 \leq t \leq 0$$

$$= 2.5(2-t); 0 \leq t \leq 2$$

$$\text{Avg value} = \frac{\text{area}}{T} = \frac{\frac{1}{2} \times 4 \times 5}{4} = 2.5$$

(2) for the given periodic signal find the T.F.S and also sketch its magnitude and phase spectrum



$$\begin{aligned} & (-1, 5) \\ & (4, 5) \\ & \frac{5 \times 1}{2} \\ & \frac{5 \times 1}{2} \\ & \frac{5}{2} = 2.5 \\ & 2.5 = 2.5 \\ & 2.5 = 2.5 \\ & 2.5 = 2.5 \end{aligned}$$

$$T = \frac{3\pi}{2} - \left(-\frac{\pi}{2}\right) = 2\pi$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1 \text{ rad/s}$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos n\omega_0 t + b_n \sin n\omega_0 t]$$

$$a_0 = \frac{\text{area}}{T} = \frac{\pi \times A}{2\pi} = A/2$$

$$a_n = \frac{2}{T} \int_0^T f(t) \cos n\omega_0 t dt$$

$$= \frac{2}{2\pi} \left[\int_{-\pi/2}^{\pi/2} A \cdot \cos nt dt + \int_{\pi/2}^{3\pi/2} 0 \cdot \cos nt dt \right]$$

$$= \frac{A}{\pi} \left[\frac{\sin nt}{n} \right]_{-\pi/2}^{\pi/2}$$

$$f(t) = A \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

$$= 0 \quad ; \quad \frac{\pi}{2} < t < \frac{3\pi}{2}$$

$$a_n = \frac{2A}{n\pi} ; n=1, 5, 9, \dots$$

$$= -\frac{2A}{n\pi} ; n=3, 7, 11, \dots$$

$$= 0 ; n = \text{even}$$

$$b_n = \frac{2}{T} \int_0^T f(t) \sin n\omega_0 t dt$$

$$= \frac{2}{2\pi} \left[\int_{-\pi/2}^{\pi/2} A \sin nt dt + \int_{\pi/2}^{3\pi/2} 0 \sin nt dt \right]$$

$$= -\frac{A}{\pi} \left[\frac{\cos nt}{n} \right]_{-\pi/2}^{\pi/2} = 0$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$$

$$= a_0 + a_1 \cos t + a_2 \cos 3t + a_3 \cos 5t + \dots$$

$$= \frac{A}{2} + \frac{2A}{\pi} (\cos t + 0 + \left(-\frac{2A}{3\pi}\right) + \dots)$$

$$= \frac{A}{2} + \frac{2A}{\pi} \left[\frac{1}{1} \cos t - \frac{1}{3} \cos 3t + \frac{1}{5} \cos 5t + \dots \right]$$

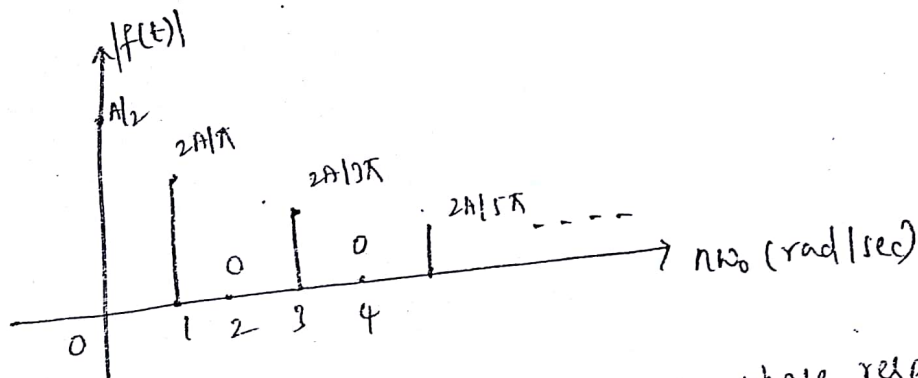
$$a_0 = A/2 ; |a_0| = \sqrt{Re^2 + Im^2} = \sqrt{\left(\frac{A}{2}\right)^2 + 0^2} = A/2$$

$$a_1 = \frac{2A}{\pi} ; |a_1| = 2A/\pi$$

$$a_2 = 0 ; |a_2| = 0$$

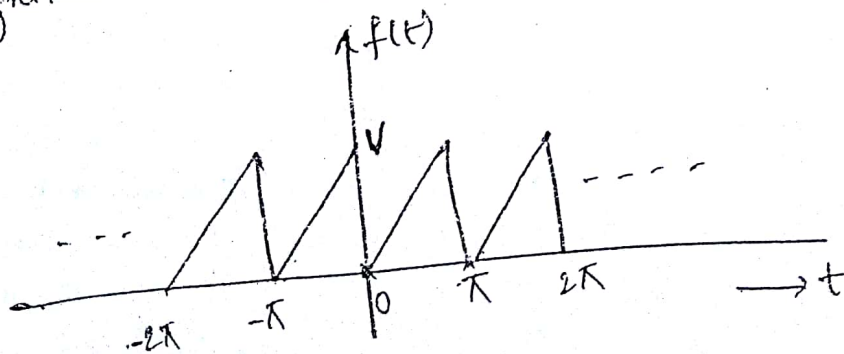
$$a_3 = \frac{-2A}{3\pi} ; |a_3| = 2A/3\pi$$

$$\angle a_0 = \tan^{-1} \left(\frac{Im}{Real} \right) = \tan^{-1} \left(0 / A/2 \right) = 0$$



$\angle f(t) \rightarrow$ TFS has always zero phase response.

(3) Find the E.F.S for the given periodic signal and sketch its magnitude and phase spectrum.



$$T = 2\pi$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1 \text{ rad/sec}$$

$$f(t) = \frac{V}{\pi} t, 0 < t < \pi$$

Ex. 1.1

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t}$$

$$c_0 = \text{avg. value} = \frac{\text{area}}{T} = \frac{\frac{1}{2} \pi V}{\pi} = V/2$$

$$c_n = \frac{1}{T} \int_0^T f(t) e^{-jn\omega t} dt$$

$$= \frac{1}{\pi} \int_0^{\pi} \frac{Vt}{\pi} e^{-jn\omega t} dt$$

$$= \frac{V}{\pi^2} \left[\int_0^{\pi} t e^{-jn\omega t} dt \right]$$

$$c_n = \frac{jV}{2n\pi}$$

Ex. 1.2

$$f(t) = \dots c_2 e^{-j4t} + c_1 e^{-j2t} + c_0 + c_1 e^{j2t} + c_2 e^{j4t} \dots$$
$$= -\frac{jV}{4\pi} e^{-j4t} - \frac{jV}{2\pi} e^{-j2t} + \frac{V}{2} + \frac{jV}{2\pi} e^{j2t} + \frac{jV}{4\pi} e^{j4t} \dots$$

Spectrum

$$c_{-2} = -\frac{jV}{4\pi}$$

$$|c_{-2}| = \frac{V}{4\pi}$$

$$c_{-1} = -\frac{jV}{2\pi}$$

$$|c_{-1}| = \frac{V}{2\pi}$$

$$c_0 = V/2$$

$$|c_0| = V/2$$

$$c_1 = \frac{jV}{2\pi}$$

$$|c_1| = V/2\pi$$

$$c_2 = \frac{jV}{4\pi}$$

$$|c_2| = V/4\pi$$

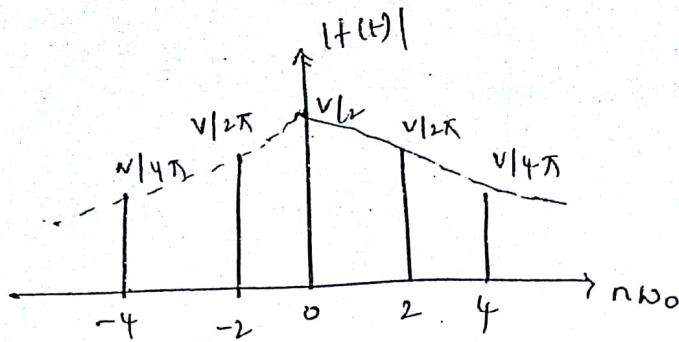
$$L_{c_2} = \tan^{-1}(-V/4\pi|0) = -\pi/2$$

$$L_{c_1} = -\pi/2$$

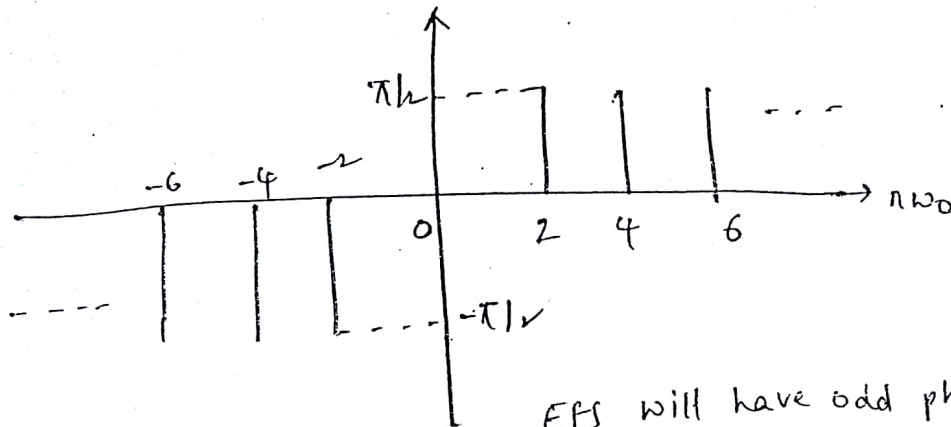
$$L_{c_0} = \tan^{-1}\left(\frac{0}{V/2}\right) = 0$$

$$L_{c_1} = \tan^{-1}\left(\frac{V/2\pi}{V/2}\right) = \pi/2$$

$$L_{c_2} = \pi/2$$



EFs have even magnitude response



EFs will have odd phase response

Spectrum analysis using Fourier series

Cosine F.S.

Remarks

T.F.S

bn

an

a-

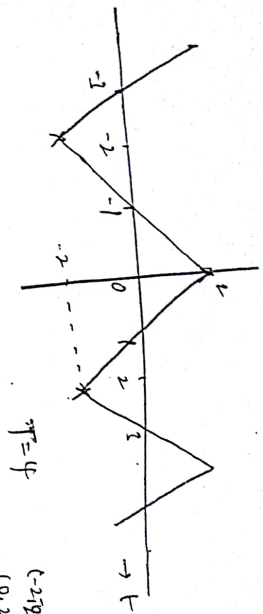
Spectrum analysis using Fourier Series

No.	Signal	Condition	Waveform	a ₀ a _n b _n	T.F.S	Remarks	Cosine F.S
1.	even signal	$f(t) = f(-t)$		✓ 0	$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t$	1. It has only even frequencies	$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t$
2.	odd signal (odd symmetry)	$f(t) = -f(-t)$		0 0 ✓	$f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega t$	2. It has only odd frequencies	$f(t) = \sum_{n=1}^{\infty} b_n \cos [n\omega t - \pi/2]$
3.	Half wave symmetry (neither even nor odd)	$f(t) = -f(t \pm T/2)$		$\begin{matrix} n(\text{even}) & \begin{cases} a_n \rightarrow 0 \\ b_n \rightarrow \checkmark \end{cases} \\ n(\text{odd}) & \begin{cases} a_n \rightarrow \checkmark \\ b_n \rightarrow 0 \end{cases} \end{matrix}$	$f(t) = \sum_{n=1}^{\infty} \begin{cases} a_n \cos n\omega t \\ b_n \sin n\omega t \end{cases}$	1. It has only odd frequencies	$f(t) = \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \cos (n\omega t - \tan^{-1}(\frac{b_n}{a_n}))$
4.	even signal (even symmetry)	$f(t) = f(-t)$ $f(t) = -f(t \pm T/2)$		0 ✓ 0	$f(t) = \sum_{n=1}^{\infty} a_n \cos n\omega t$	4. It has odd frequencies	It has only odd coefficients $f(t) = \sum_{n=1}^{\infty} a_n \cos n\omega t$

$$f(t) = a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \cos \left[n\omega t - \tan^{-1} \left(\frac{b_n}{a_n} \right) \right]$$

* find the Trigno and Exp. Fourier series for the periodic signal and also sketch their magnitude and phase spectrum.

Sol:



$$f(t) = \begin{cases} 4t & ; 0 < t < 2 \\ -4t & ; -2 < t < 0 \end{cases}$$

$$y_0 = \frac{1}{T} \int_0^T f(t) dt$$

$$\Rightarrow f(t) = 4t + 2 \quad f(t) = 2(t+1)$$

$$(t+2), (0, 2)$$

$$y_{t+2} = \frac{1}{2} (2-2)$$

$$y_{t+2} = -2t - 4$$

$$y = -2t - 2$$

$$y = -2(t-1)$$

$$f(t) = 2(t-1)$$

$$y_0 = \frac{2T}{T} = \frac{2T}{4} = T/2$$

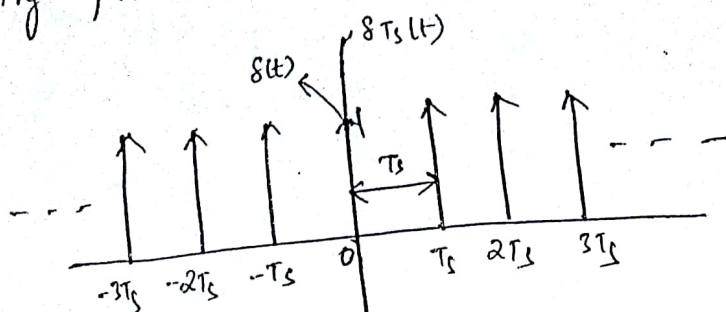
$$a_0 = 0$$

$$a_n = \frac{2}{T} \int_0^T f(t) \cos n\omega_0 t dt$$

$$= \frac{1}{2} \left[\int_{-2}^0 2(t+1) \cos n\left(\frac{\pi}{2}\right)t \, dt + \int_0^2 -2(t-1) \cos n\left(\frac{\pi}{2}\right)t \, dt \right]$$

$$= \frac{1}{2} \left[2(t+1) \sin n\left(\frac{\pi}{2}\right)t \times \frac{2}{n\pi} \right]$$

** Find the exp. f.s of the "periodic impulse train ($\delta_{Ts}(t)$)" or "Sampling function" or "Comb Function"



E.F.S

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$C_n = \frac{1}{T} \int_0^T f(t) e^{-jn\omega_0 t} dt$$

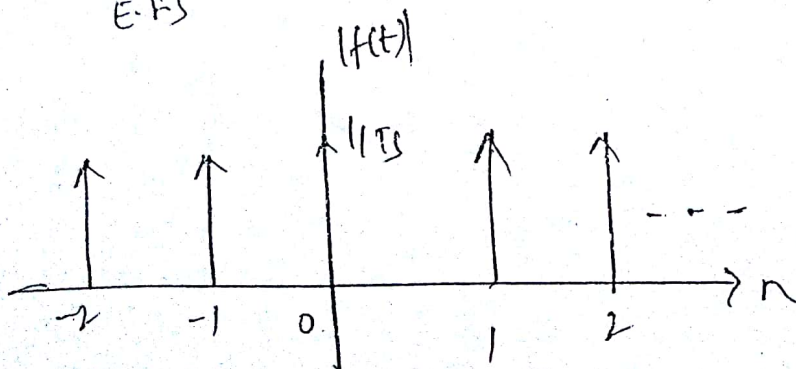
$$T = T_s ; \quad \omega = 2\pi / T_s$$

$$C_n = \frac{1}{T_s} \int_0^{T_s} \delta(t) \cdot e^{-jn(\frac{2\pi}{T_s})t} dt$$

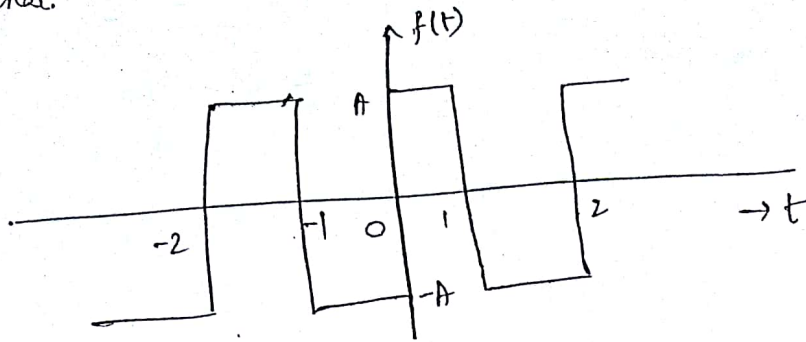
$$C_n = \frac{1}{T_s}$$

$$f(t) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} e^{jn(\frac{2\pi}{T_s})t}$$

E.F.S



Find the fourier series for the co-eff for the given periodic signal.



$$f(t) = A ; 0 < t < 1$$

$$-A ; -1 < t < 0$$

$$T = 2$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi \text{ rad/sec}$$

$$C_n = \frac{1}{T} \int_0^T f(t) e^{-jn\omega t} dt$$

$$= \frac{1}{2} \left[\int_0^1 A e^{-jn\pi t} dt + \int_{-1}^0 -A e^{-jn\pi t} dt \right]$$

$$= \frac{1}{2} \left[A \cdot \frac{e^{-jn\pi t}}{-jn\pi} \Big|_0^1 + -A \frac{e^{-jn\pi t}}{-jn\pi} \Big|_{-1}^0 \right]$$

$$= \frac{1}{2} \left[\frac{A}{jn\pi} \left[-e^{-jn\pi} + 1 \right] + \frac{A}{jn\pi} \left[-1 + e^{-jn\pi} \right] \right]$$

$$= \frac{1}{2} \left[\frac{2A}{jn\pi} - \frac{A}{jn\pi} \left[e^{-jn\pi} + e^{jn\pi} \right] \right]$$

$$= \frac{A}{jn\pi} - \frac{A}{jn\pi} \cos n\pi \quad \text{2) } -\frac{A}{n\pi} + j \frac{A}{n\pi} (-1)^n$$

$$= \frac{A}{jn\pi} \left[1 - (-1)^n \right] \quad \text{32}$$

31/11/23

* Fourier Transform: The transformation of non-periodic signals from time domain to frequency domain

→ Continuous Fourier Transform

$$F[x(t)] = X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt.$$

or
 $X(\omega)$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt.$$

→ Inverse Fourier Transform

$$F^{-1}[X(j\omega)] = x(t)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega.$$

$$x(t) = \int_{-\infty}^{\infty} X(f) \cdot e^{j2\pi ft} df.$$

→ Convergence of Fourier Transform

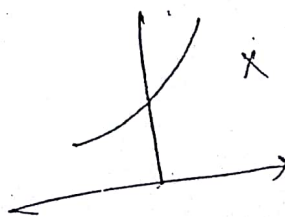
Dirichlet's conditions

(1) $x(t)$ must be non-periodic

(2) $x(t)$ must be absolutely integrable

i.e. $\int_{-\infty}^{\infty} x(t) dt < \infty$

i.e. $x(t)$ must be a bounded signal



* Spectrum

Note:

(1) The

(2)

Ans

Sol:-

* Spectrum: ~~the~~

Note:

- (1) The spectrum of Fourier transform is always a continuous frequency response
- (2) Its magnitude is always a symmetrical response and phase is always anti-symmetrical (odd) response

$$X(j\omega) = |X(j\omega)| e^{j \angle X(j\omega)}$$

Find the Fourier transform of $x(t) = e^{-at} u(t)$, and also sketch its magnitude and phase spectrum

Sol:-

$$x(t) = e^{-at} u(t) \quad (a > 0)$$

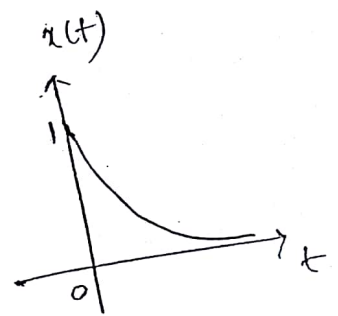
$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t} dt$$

$$= \int_0^{\infty} e^{-(a+j\omega)t} dt$$

$$= \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right]_0^{\infty} = \frac{1}{a+j\omega}$$

$$X(j\omega) = \frac{1}{a+j\omega}$$

$$= \frac{1}{\sqrt{a^2 + \omega^2}} e^{-j \tan^{-1}(\omega/a)}$$



$$|X(\omega)| = \sqrt{Re^2 + Im^2}$$

$$X(\omega) = \frac{1}{a+j\omega} \times \frac{a-j\omega}{a-j\omega}$$

$$= \frac{a-j\omega}{a^2 + \omega^2} = \underbrace{\frac{a}{a^2 + \omega^2}}_{\text{Real}} - j \underbrace{\left[\frac{\omega}{a^2 + \omega^2} \right]}_{\text{Imag}}$$

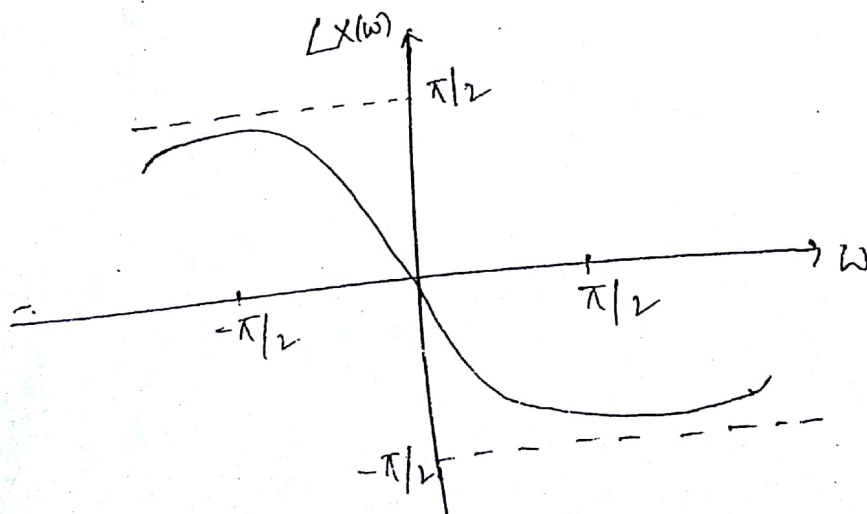
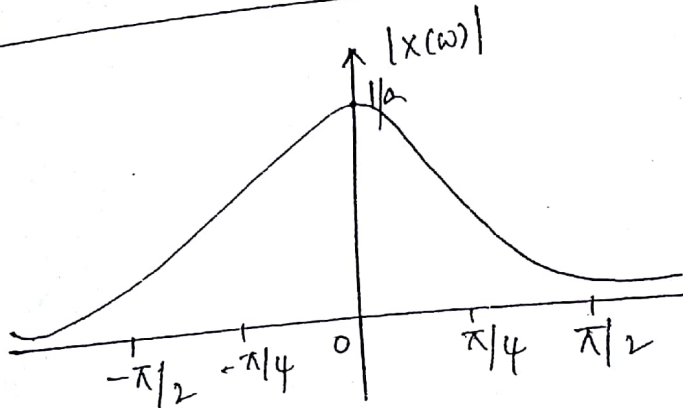
$$|X(\omega)| = \sqrt{\left(\frac{a}{a^2 + \omega^2}\right)^2 + \left(\frac{-\omega}{a^2 + \omega^2}\right)^2}$$

$$= \sqrt{\frac{a^2 + \omega^2}{(a^2 + \omega^2)^2}} = \frac{1}{\sqrt{a^2 + \omega^2}}$$

$$\angle X(\omega) = \tan^{-1}\left(\frac{\text{Im}}{\text{Real}}\right)$$

$$= \tan^{-1}\left[\frac{\frac{-\omega}{a^2 + \omega^2}}{\frac{a}{a^2 + \omega^2}}\right]$$

$$\boxed{\angle X(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)}$$



$$x(t) = te^{-at} u(t) \quad [a > 0]$$

$$X(j\omega) = \int_{-\infty}^{\infty} te^{-at} u(t) e^{-j\omega t} dt = \int_0^{\infty} te^{-at} e^{-j\omega t} dt$$

$$= \int_0^{\infty} t [e^{-t(a+j\omega)}] dt$$

$$= \frac{t e^{-t(a+j\omega)}}{a+j\omega} + \int \frac{e^{-t(a+j\omega)}}{a+j\omega}$$

$$= \frac{t e^{-t(a+j\omega)}}{a+j\omega} + \left[\frac{e^{-t(a+j\omega)}}{(a+j\omega)^2} \right]_0^{\infty}$$

$$= \frac{1}{(a+j\omega)^2}$$

$$|X(\omega)| = \sqrt{\text{Re}^2 + \text{Im}^2}$$

$$[X(\omega)] = \frac{1}{(a+j\omega)^2} \times \frac{(a-j\omega)^2}{(a-j\omega)^2}$$

$$= \frac{a^2 - \omega^2 - 2aj\omega}{(a^2 - \omega^2)^2 - 4a^2\omega^2}$$

By conjugate property

$$|X(\omega)| = \sqrt{X(\omega) \cdot X^*(\omega)}$$

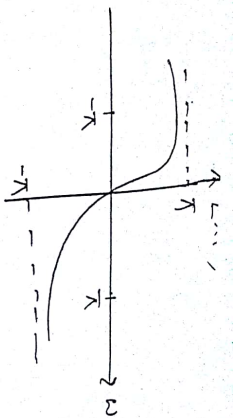
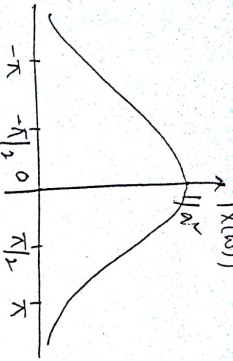
$$= \sqrt{\frac{1}{(a+j\omega)^2} \times \frac{1}{(a-j\omega)^2}}$$

$$\Rightarrow \boxed{|X(\omega)| = \frac{1}{a^2 + \omega^2}}$$

$$\Rightarrow X(\omega) = \frac{1+j0}{(a+j\omega)^2}$$

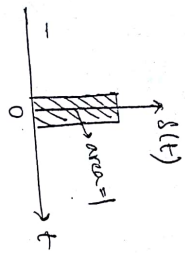
$$= \tan^{-1}\left(\frac{0}{1}\right) - 2 \tan^{-1}\left(\frac{\omega}{a}\right)$$

$$\boxed{\angle X(\omega) = -2 \tan^{-1}\left(\frac{\omega}{a}\right)}$$



* find the fourier transform of $[g(t)]$ impulse function

Soln



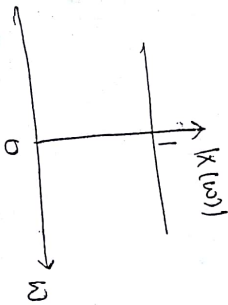
$$X(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt$$

$$[X(0) = 1]$$

$$\Rightarrow |X(\omega)| = 1$$

No phase response

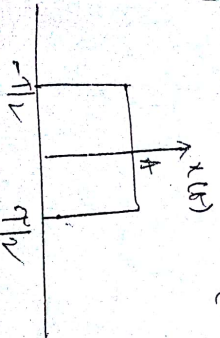
$$\angle X(\omega) = 0$$



* find the fourier transform of gate function

Gate function $A\tau(t/\tau)$

Rectangular Function $A\text{rect}(t/\tau)$



$$x(t) = A; \quad -\tau/2 \leq t \leq \tau/2$$

$$\begin{aligned}
 X(\omega) &= \int_{-T/2}^{T/2} A e^{-j\omega t} dt \\
 &= A \cdot \left[\frac{e^{-j\omega t}}{-j\omega} \right]_{-T/2}^{T/2} \\
 &= 2 \cdot \frac{A}{\omega} \left[\frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2j} \right]
 \end{aligned}$$

$$X(\omega) = \frac{2A}{\omega} \sin\left(\frac{\omega T}{2}\right)$$

$$\rightarrow \text{sinc} x = \frac{\sin \pi x}{\pi x}$$

$$\rightarrow \text{sax} = \frac{\sin x}{x}$$

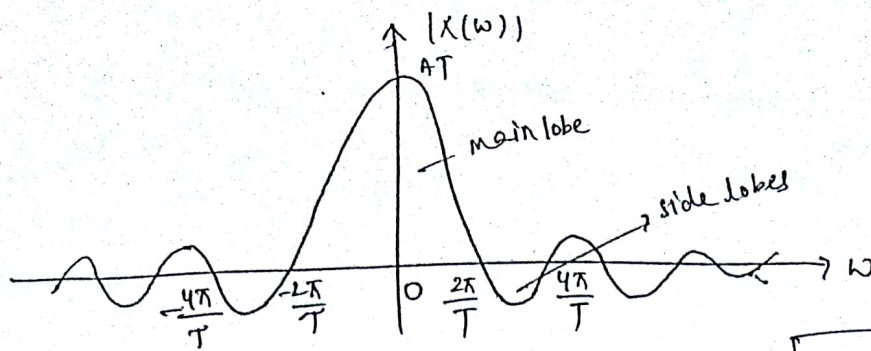
$$X(\omega) = \frac{2A}{\omega} \frac{\sin(\omega T/2)}{\omega T/2} \times \omega T/2$$

$$X(\omega) = A T \cdot \text{sax}(\omega T/2)$$

$$X(f) = \frac{2A}{2\pi f} \frac{\sin\left(\frac{2\pi f T}{2}\right)}{\pi f T} \times \pi f T$$

$$X(f) = A T \cdot \text{sinc}(f T)$$

$$\left. \begin{array}{l} A \pi(t/T) \text{ or} \\ A \text{rect}(t/T) \end{array} \right\} \begin{array}{l} \longleftrightarrow A T \text{sax}(\omega T/2) \\ \longleftrightarrow A T \text{sinc}(f T) \end{array}$$



$$\text{Theoretical B.W} = \frac{2\pi}{T} - \left(-\frac{2\pi}{T}\right) = \frac{4\pi}{T}$$

$$\text{Practical B.W} = \frac{2\pi}{T}$$

70-75% — main lobe
20-25% — side lobes

* Band Width: The range of frequencies where most of the spectrum is concentrated

* Rise time (tr)

$$* \boxed{tr \propto \frac{1}{\text{B.W}}}$$

find the FT of $\frac{1}{a+jt}$

Soln- $x(t) = \frac{1}{a+jt}$

$$X(j\omega) = \int_{-\infty}^{\infty} \frac{1}{a+jt} e^{j\omega t} dt$$

$$e^{-at} u(t) \leftrightarrow \frac{1}{a+j\omega}$$

By duality

$$\frac{1}{a+jt} \leftrightarrow 2\pi e^{-a(-\omega)} u(-\omega)$$

$$\leftrightarrow 2\pi e^{a\omega} u(-\omega)$$

* Prof.

(1) SI

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* Properties of Fourier Transform:

(i) symmetric property or property of duality

$$\text{If } x(t) \xleftrightarrow{\text{F.T.}} X(\omega) \text{ then}$$

$$X(t) \longleftrightarrow 2\pi x(-\omega)$$

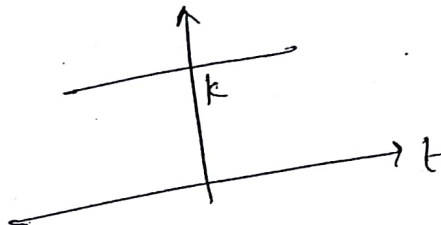
Proof $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$

$$2\pi x(t) = \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t) e^{-j\omega t} dt$$

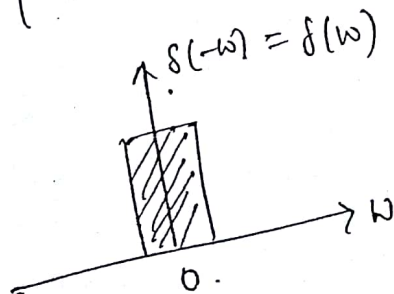
$$\boxed{X(t) \longleftrightarrow 2\pi x(-\omega)}$$

(ii) Fourier transform of $x(t) = k$



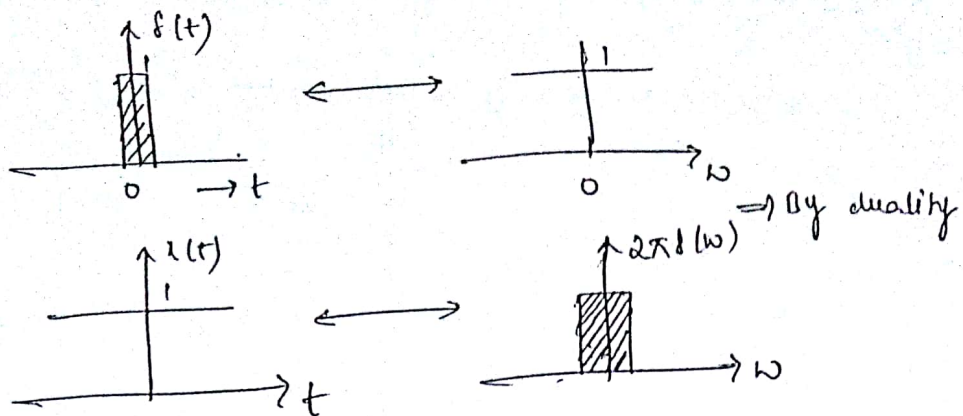
$$f(t) \longleftrightarrow 1$$

$$1 \longleftrightarrow 2\pi f(-\omega)$$



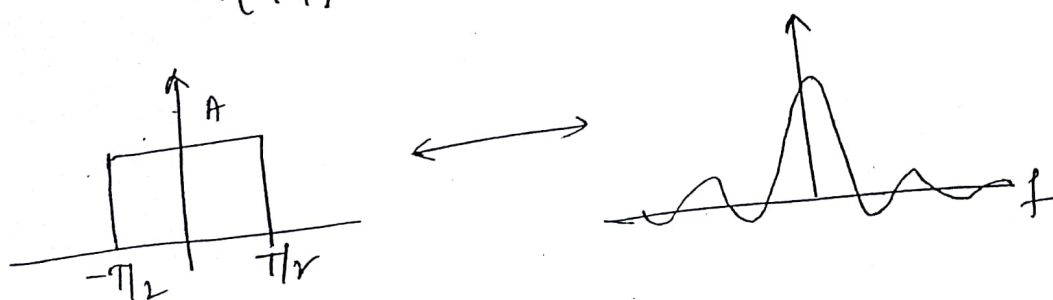
$$1 \longleftrightarrow 2\pi f(\omega)$$

$$\boxed{k \longleftrightarrow 2\pi k f(\omega)}$$

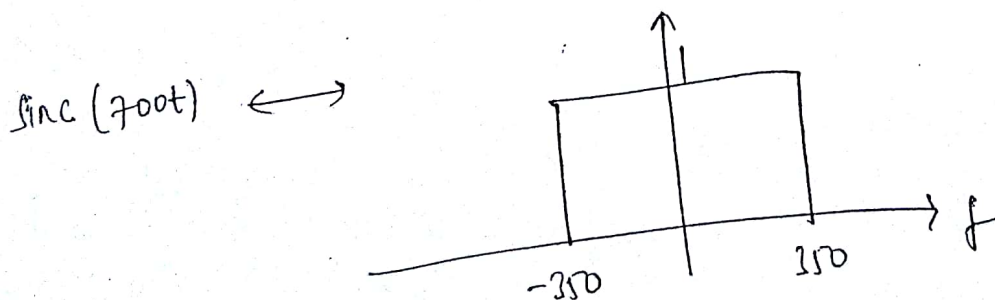
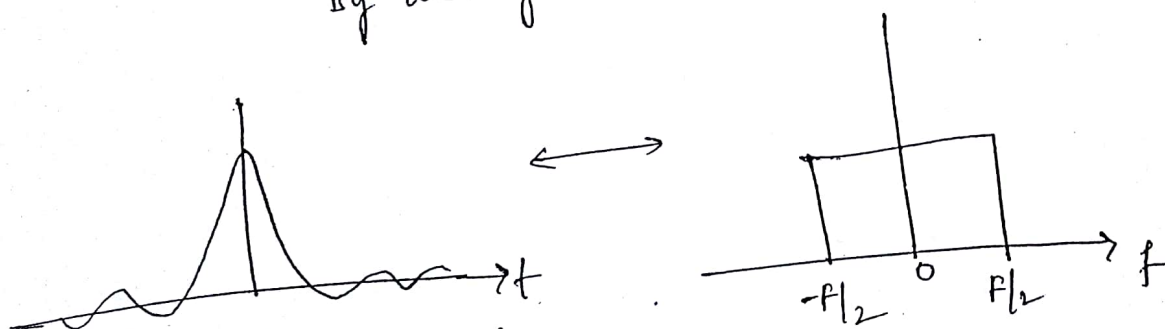


12) Fourier transform of $x(t) = \text{sinc}(700t)$

$$A\pi(t/T) \leftrightarrow A\text{sinc}(fT)$$



By duality



13) for
Sol:-

4/2/13

Pro

Problem

1)

Q3) Fourier transform of $\pi(t)$

Sol:-

$$x(t) = \pi(t) \longleftrightarrow A T \text{sinc}(fT)$$

$$\boxed{\pi(t) \longleftrightarrow \text{sinc}(f)}$$

4/2/13

Property of Linearity

$$\text{If } x_1(t) \longleftrightarrow X_1(\omega)$$

$$\text{and } x_2(t) \longleftrightarrow X_2(\omega) \text{ then}$$

$$ax_1(t) + bx_2(t) \longleftrightarrow aX_1(\omega) + bX_2(\omega)$$

Proof $\mathcal{F}[ax_1(t) + bx_2(t)] = \int_{-\infty}^{\infty} [ax_1(t) + bx_2(t)] e^{-j\omega t} dt$

$$= a \int_{-\infty}^{\infty} x_1(t) e^{-j\omega t} dt + b \int_{-\infty}^{\infty} x_2(t) e^{-j\omega t} dt$$
$$= aX_1(\omega) + bX_2(\omega)$$

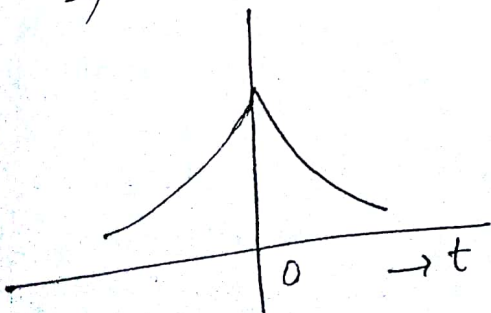
Problems

1) $x(t) = e^{-a|t|}$

$$= e^{-at} u(t) + e^{at} u(-t)$$

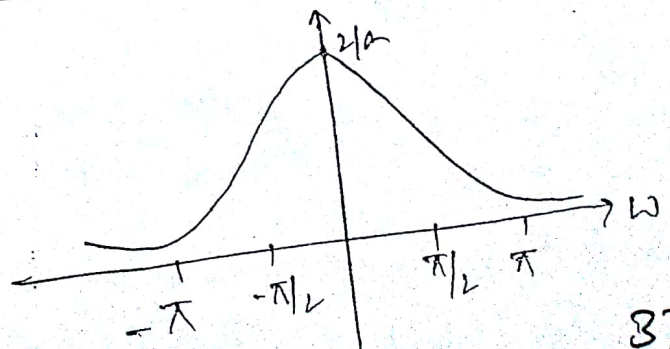
$$= \frac{1}{a+j\omega} + \frac{1}{a-j\omega}$$

$$\Rightarrow a > 0$$



$$\boxed{X(\omega) = \frac{2a}{a^2 + \omega^2}}$$

$$\Rightarrow |X(\omega)| = \frac{2a}{a^2 + \omega^2}$$



(2) $x(t) = e^{-at} \text{sgn}(t)$

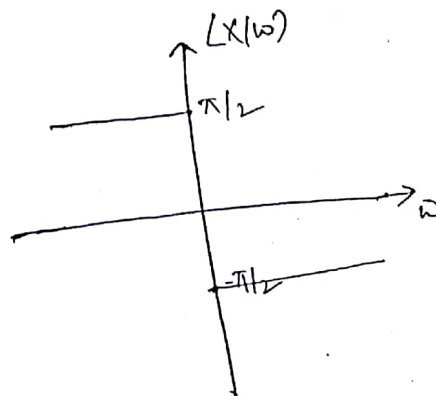
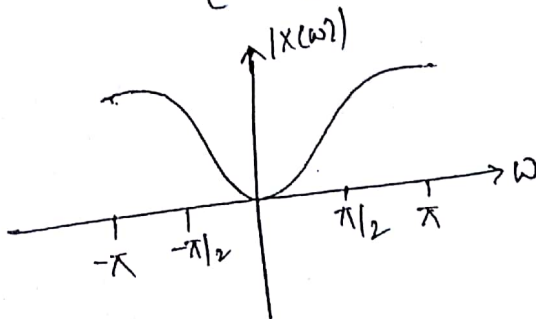
Sol:-

$$x(t) = e^{-at} \text{sgn}(t) \\ = e^{-at} u(t) - e^{at} u(-t)$$

$$X(\omega) = \frac{1}{a+j\omega} - \frac{1}{a-j\omega}$$

$$X(\omega) = \frac{-2j\omega}{[a^2 + \omega^2]}$$

$$|X(\omega)| = \left[\frac{2\omega}{a^2 + \omega^2} \right] \quad \angle X(\omega) = \pm \pi/2$$



Time shifting Property

If $x(t) \leftrightarrow X(\omega)$
then $x(t-t_0) \leftrightarrow X(\omega)e^{-j\omega t_0}$

$$= \int_{-\infty}^{\infty} x(t-t_0) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(a) e^{-j\omega a} da e^{-j\omega t_0}$$

$$= x(a) e^{-j\omega t_0}$$

$$= \cancel{x(a)} e^{-j\omega t_0} = X(\omega) e^{-j\omega t_0}$$

* find
y(t)

Sol:-

* 1

Sol:-

$$\begin{aligned} & \delta(t-t_0) \\ & \xrightarrow{\text{FT}} e^{-j\omega t_0} \\ & \frac{1}{t-a+t_0} \end{aligned}$$

* find the frequency response for the given response

$$y(t) = x(t+2) - x(t-2)$$

Sol:-

$$y(t) = x(t+2) - x(t-2)$$

F.T on b-s

$$Y(\omega) = [e^{j\omega(2)} - e^{-j\omega(2)}] X(\omega)$$

$$\frac{Y(\omega)}{X(\omega)} = 2j \left[\frac{e^{+2j\omega} - e^{-2j\omega}}{2j} \right]$$

$$= 2j \frac{\sin(2\omega)}{2\omega} \times 2\omega$$

$$\boxed{\frac{Y(\omega)}{X(\omega)} = 4j\omega \text{sinc}(2\omega)}$$

* $x(t) = e^{-2t} u(t-3)$

Sol:-

$$x(t) = e^{-2t} u(t-3)$$

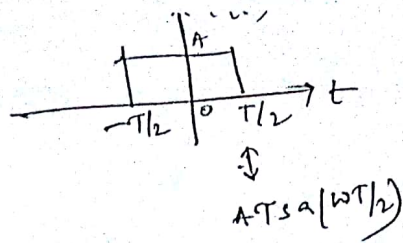
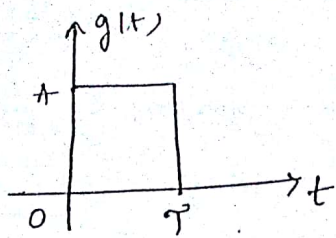
$$e^{-2t} u(t) \longleftrightarrow \frac{1}{2+j\omega}$$

$$e^{-2(t-3)} u(t-3) \longleftrightarrow e^{-j\omega(3)} \cdot \frac{1}{(2+j\omega)}$$

$$e^{-2t} u(t-3) = e^{-2(t-3)} u(t-3) \times e^{-6}$$

$$= \frac{e^{-3j\omega}}{(2+j\omega)} \times e^{-6}$$

$$= \frac{e^{-3(2+j\omega)}}{(2+j\omega)}$$



$$g(t) = x(t - T/2)$$

$$G(\omega) = e^{-j\omega T/2} \times A T \text{sinc}(\omega T/2)$$

Time scaling property

If $x(t) \leftrightarrow X(\omega)$ then
 $x(at) \leftrightarrow \frac{1}{|a|} X(\omega/a)$

$$= \int_{-\infty}^{\infty} x(at) \cdot e^{-j\omega t} \cdot dt$$

Let $at = k$
 $dt = dk/a$

$$= \int_{-\infty}^{\infty} x(k) e^{-j\omega(k/a)} \cdot dk/a$$

$$= \frac{1}{a} X(\omega/a)$$

* x/s $y(t) = x(5t) + x(t/5)$

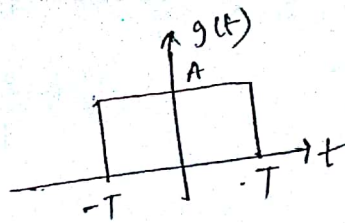
$$= \int_{-\infty}^{\infty} x(5t) e^{-j\omega t} dt + \int_{-\infty}^{\infty} x(t/5) e^{-j\omega t} dt$$

$$= \frac{1}{5} X(\omega/5) + 5X(5\omega)$$

* $x(5t+3) = x[5(t+3/5)]$

$$= \frac{1}{5} e^{j\omega(3/5)} X(\omega/5)$$

*

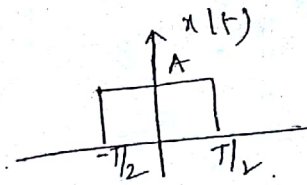


$$g(t) = x(T/2)$$

$$x(T) \leftrightarrow A T \text{sinc}(\omega T/2)$$

$$= \frac{1}{T/2} A T \text{sinc}((\omega T/2)/2)$$

$$G(\omega) = 2 A T \text{sinc}(\omega T)$$



Shifting in frequency domain

$$\text{If } x(t) \leftrightarrow X(\omega)$$

$$e^{j\omega_0 t} x(t) \leftrightarrow X(\omega - \omega_0)$$

Proof

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega - \omega_0) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(k) e^{jkt} e^{j\omega_0 t} dk$$

$$= \underline{\underline{e^{j\omega_0 t} x(t)}}$$

$$\omega - \omega_0 = k$$

$$k = \omega - \omega_0$$

* $x(t) = \cos \omega_c t$

Soln

$$x(t) = \cos \omega_c t$$

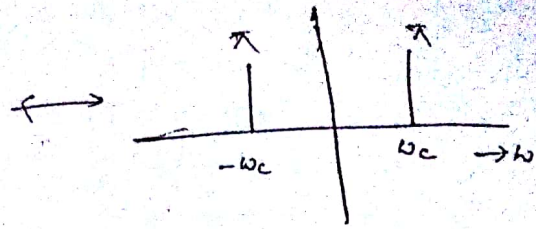
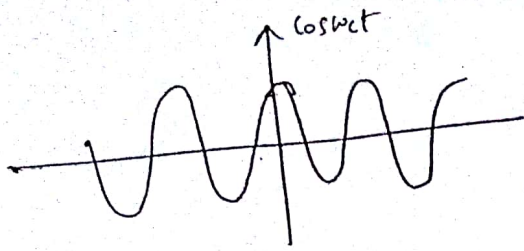
$$= \frac{1}{2} (e^{j\omega_c t} + e^{-j\omega_c t})$$

$$= \frac{[2\pi \delta(\omega - \omega_c) + 2\pi \delta(\omega + \omega_c)]}{2}$$

$$1 \leftrightarrow 2\pi \delta(\omega)$$

$$1x e^{j\omega_c t} \leftrightarrow 2\pi \delta(\omega - \omega_c)$$

$$\cos \omega_c t \longleftrightarrow \pi [\delta(\omega + \omega_c) + \delta(\omega - \omega_c)]$$



$$x(t) = \sin \omega_c t$$

$$\sin \omega_c t = \frac{e^{j\omega_c t} - e^{-j\omega_c t}}{2j}$$

$$X(\omega) = \frac{1}{2j} [1 \times e^{j\omega_c t} - 1 \times e^{-j\omega_c t}]$$

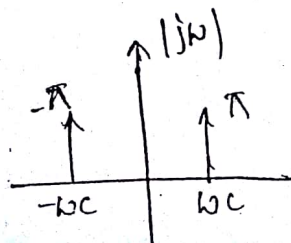
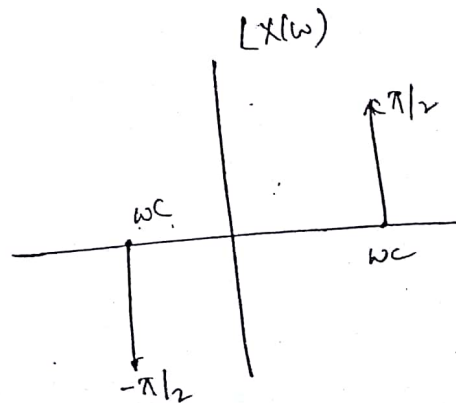
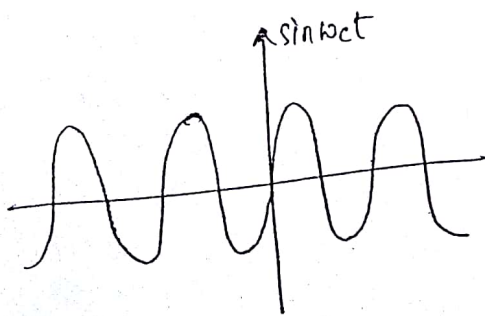
$$= \frac{1}{2j} [2\pi \delta(\omega - \omega_c) e^{j\omega_c t} - 2\pi \delta(\omega + \omega_c) e^{-j\omega_c t}]$$

$$= \frac{\pi}{j} [\delta(\omega - \omega_c) e^{j\omega_c t} - \delta(\omega + \omega_c) e^{-j\omega_c t}]$$

$$= \frac{\pi}{j} [\delta(\omega - \omega_c) - \delta(\omega + \omega_c)]$$

$$X(\omega) = \pi [\delta(\omega + \omega_c) - \delta(\omega - \omega_c)]$$

$$|X(\omega)| = \pm \pi/2$$



Differentiation in Time domain:

$$\text{If } x(t) \leftrightarrow X(\omega)$$

$$\text{then } \frac{d}{dt} x(t) \leftrightarrow j\omega X(\omega)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$\frac{d}{dt} x(t) = \frac{d}{dt} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \right]$$

$$= j\omega \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \right]$$

$$= j\omega X(\omega)$$

$$\boxed{\frac{d^n}{dt^n} x(t) \leftrightarrow (j\omega)^n X(\omega)}$$

* The magnitude and phase response of a system defined as

$$y(t) = 2 \frac{d}{dt} x(t-3)$$

$$\text{So/If } y(t) = 2 \frac{d}{dt} x(t-3)$$

$$Y(\omega) = 2[j\omega] e^{-3j\omega} X(\omega)$$

$$\frac{Y(\omega)}{X(\omega)} = H(\omega) = 2j\omega e^{-3j\omega}$$

$$|H(\omega)| = |2j\omega| |e^{-3j\omega}|$$

$$|H(\omega)| = 2\omega$$

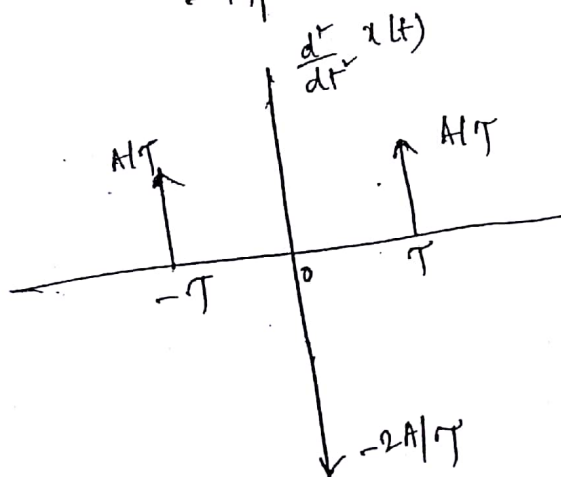
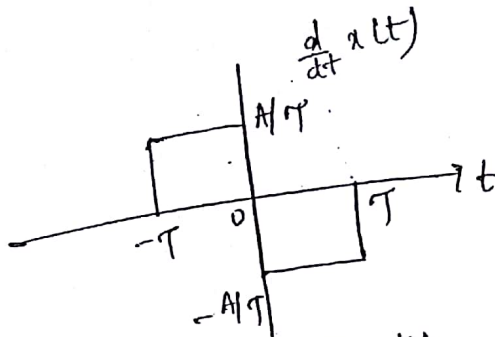
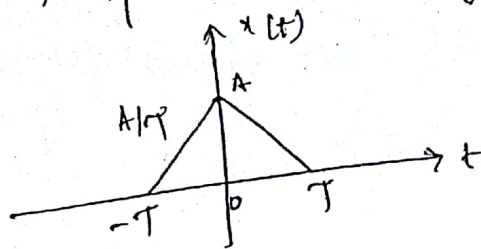
$$\angle H(\omega) = 2j\omega \cdot e^{-3j\omega}$$

$$= \pi/2 + (-3\omega)$$

$$\text{and } \pi/2$$

* find the F.T of standard triangular function

Sol:



$$\frac{d^2}{dt^2} x(t) = \frac{A}{T} \delta(t+T) - \frac{2A}{T} \delta(t) + \frac{A}{T} \delta(t-T)$$

(P.T on b.s)

$$(\omega)^2 X(\omega) = 2 \cdot \frac{A}{T} \left[\frac{e^{j\omega T} + e^{-j\omega T}}{2} \right] - \frac{2A}{T}$$

$$-\omega^2 X(\omega) = \frac{2A}{T} [\cos \omega T - 1]$$

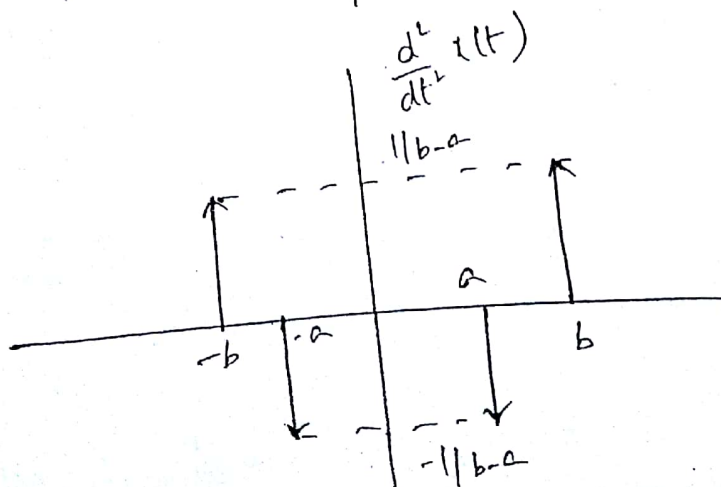
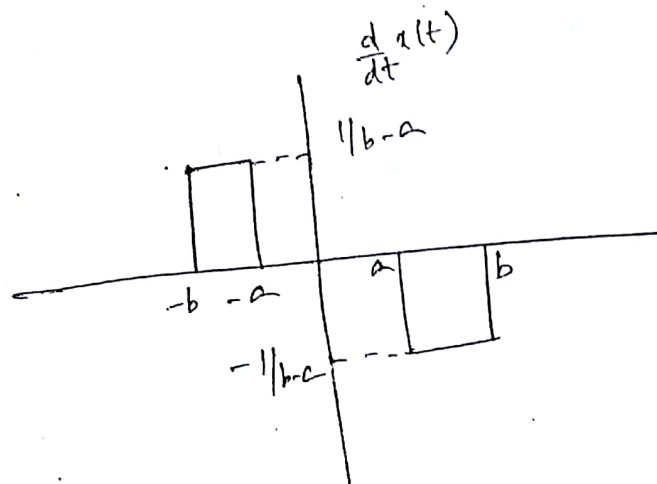
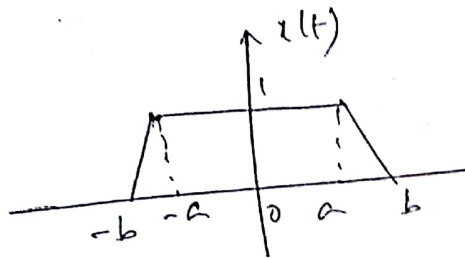
$$\omega^2 X(\omega) = \frac{2A}{T} [1 - \cos \omega T]$$

$$\omega^2 X(\omega) = \frac{2A}{T} \times \frac{2 \sin^2(\omega T/2)}{(\omega T/2)^2} \times \frac{\omega^2 T^3}{4}$$

$$X(\omega) = AT \text{sinc}^2(\omega T/2)$$

$$A \Delta(t/T) \leftrightarrow AT \text{sinc}^2(\omega T/2) \\ \leftrightarrow AT \text{sinc}^2(fT)$$

*



$$\frac{d^2}{dt^2} z(t) = \frac{1}{(b-a)} \left[\delta(t+b) - \delta(t+a) - \delta(t-a) + \delta(t-b) \right]$$

f-T m b-5

$$(j\omega)^n X(\omega) = \frac{1}{b-a} \left[\frac{(e^{j\omega b} + e^{-j\omega b})}{2} - \frac{(e^{j\omega a} + e^{-j\omega a})}{2} \right]$$

$$-j\omega X(\omega) = \frac{2}{b-a} [\cos \omega b - \cos \omega a]$$

$$X(\omega) = \frac{2}{(a-b)\omega^2} [\cos \omega b - \cos \omega a]$$

Differentiation in frequency domain:

$$t x(t) \leftrightarrow X(\omega)$$

$$-jt x(t) \leftrightarrow \frac{d}{d\omega} X(\omega)$$

Proof:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\frac{d}{d\omega} X(\omega) = -jt \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\boxed{t x(t) \leftrightarrow j \frac{d}{d\omega} X(\omega)}$$

$$* x(t) = t \cdot e^{-at} u(t)$$

$$\text{Soln: } e^{-at} u(t) \leftrightarrow \frac{1}{a+j\omega}$$

$$t \cdot e^{-at} u(t) \leftrightarrow j \frac{d}{d\omega} \left[\frac{1}{a+j\omega} \right]$$

$$\leftrightarrow j \left[\frac{-j}{(a+j\omega)^2} \right]$$

$$\leftrightarrow \frac{1}{(a+j\omega)^2}$$

$$t^2 \cdot e^{-at} u(t) \leftrightarrow j \cdot \frac{d}{d\omega} \left(\frac{1}{(a+j\omega)^2} \right)$$

$$\leftrightarrow j \cdot \frac{-2j}{(a+j\omega)^3} = \frac{2}{(a+j\omega)^3}$$

$$\boxed{t^n \cdot e^{-at} u(t) \leftrightarrow \frac{n!}{[a+j\omega]^{n+1}}}$$

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** Convolution in time domain property

$$\text{If } x_1(t) \leftrightarrow x_1(\omega)$$

$$x_2(t) \leftrightarrow x_2(\omega) \text{ then}$$

$$\boxed{x_1(t) * x_2(t) \leftrightarrow x_1(\omega) \cdot x_2(\omega)}$$

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

$$\mathcal{F}[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \right] e^{-j\omega t} dt$$

$$\begin{aligned}
 &= \int_{-\infty}^{\infty} x_1(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau) e^{j\omega t} dt \right] d\tau \\
 &= \int_{-\infty}^{\infty} x_1(\tau) \left[e^{-j\omega\tau} H(\omega) \right] d\tau \\
 &= H(\omega) \int_{-\infty}^{\infty} x_1(\tau) e^{-j\omega\tau} d\tau \\
 &= H(\omega) \cdot X(\omega)
 \end{aligned}$$

Problem

(i) Given that $x_1(t) = e^{-t} u(t)$, $x_2(t) = e^{2t} u(t)$ then the o/p response $y(t) = x_1(t) * x_2(t)$ is

Soln

$$\begin{aligned}
 x_1(t) &= e^{-t} u(t) \\
 x_2(t) &= e^{2t} u(t)
 \end{aligned}$$

Ans

$$\begin{aligned}
 y(t) &= x_1(t) * x_2(t) \\
 y(\omega) &= X_1(\omega) \times X_2(\omega) \\
 Y(\omega) &= \frac{1}{(1+j\omega)} \times \frac{1}{(2+j\omega)}
 \end{aligned}$$

$$= \frac{A}{(1+j\omega)} + \frac{B}{(2+j\omega)}$$

$$\begin{aligned}
 \#; \quad &1+j\omega=0 \\
 &[\omega=-1]
 \end{aligned}$$

$$= \frac{1}{1+j\omega} - \frac{1}{2+j\omega}$$

$$\boxed{y(t) = e^{-t} u(t) - e^{2t} u(t)}$$

12) Find the o/p response when $x(t) = e^{-at}u(t)$ convolved with itself.

Sol:-

$$x(t) = e^{-at}u(t)$$

$$y(t) = x(t) * x(t)$$

$$Y(\omega) = \frac{1}{(a+j\omega)} \times \frac{1}{(a+j\omega)}$$

$$Y(\omega) = \frac{1}{(a+j\omega)^2}$$

$$\boxed{y(t) = t \cdot e^{-at}u(t)}$$

13) Given that $y(t) = x(t) * h(t)$, $g(t) = x(3t) * h(3t)$ and also given that $g(t) = Ay(0t)$ then the values A, B are

Sol:-

$$y(t) = x(t) * h(t), \quad g(t) = x(3t) * h(3t)$$

$$Y(\omega) = X(\omega) \times H(\omega), \quad G(\omega) = \frac{1}{3} X(\omega/3) \times \frac{1}{3} H(\omega/3)$$

$$= \frac{1}{9} X(\omega/3) \times H(\omega/3)$$

$$= \frac{1}{9} Y(\omega/3) \quad \text{--- (1)}$$

$$\Rightarrow g(t) = Ay(0t)$$

$$G(\omega) = \frac{A}{B} Y\left(\frac{\omega}{B}\right) \quad \text{--- (2)}$$

$$\boxed{B=3}; \quad \frac{A}{B} = \frac{1}{9} \quad [\text{from (1)}]$$

$$\boxed{A=1/3}$$

* Convolution in frequency domain (Modulation property)

If $x_1(t) \leftrightarrow X_1(\omega)$
 $x_2(t) \leftrightarrow X_2(\omega)$ then

$$x_1(t) \cdot x_2(t) \leftrightarrow \frac{1}{2\pi} [X_1(\omega) * X_2(\omega)]$$

Proof $X_1(\omega) * X_2(\omega) = \int_{-\infty}^{\infty} X_1(\lambda) X_2(\omega - \lambda) d\lambda$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} X_1(\lambda) X_2(\omega - \lambda) e^{-j\omega t} d\lambda \right] d\omega$$

$$= \frac{1}{(2\pi)} \int_{-\infty}^{\infty} X_1(\lambda) \left[\int_{-\infty}^{\infty} X_2(\omega - \lambda) e^{-j\omega t} d\omega \right] d\lambda$$

$$= \frac{1}{(2\pi)} \int_{-\infty}^{\infty} X_1(\lambda) X_2(\omega - \lambda) d\lambda \cdot e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\lambda) \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} X_2(\omega - \lambda) e^{j\omega t} d\omega \cdot d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\lambda) \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} X_2(u) e^{j(u+\lambda)t} du \cdot d\lambda$$

$\omega - \lambda = u$
 $\omega = u + \lambda$
 $d\omega = du$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\lambda) e^{j\lambda t} d\lambda \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} X_2(u) \cdot e^{jut} du$$

$$= \underline{\underline{x_1(t) \cdot x_2(t)}}$$

* Energy Spectral Density (ESD):

[Rayleigh's Energy Theorem]

Statement: It states that even if the transformation of the signal is taking place the energy in the signal keeps constant.

Defined as:
$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Proof:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot x(t) dt$$

$$= \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \right] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \left[\int_{-\infty}^{\infty} x(t) e^{j\omega t} dt \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

* Power Spectral Density (P.S.D) [Parsvels Power Theorem]

Avg. power $P = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) \cdot d\omega$

Where
$$S(\omega) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

i.e. P.S.D of the signal $x(t)$

$X(\omega) \rightarrow$ F.T of the signal $x(t)$.

Parsvels Relation: For periodic signals it states that even if the signal is expanded as the fourier series expansion, the power in the signal remains constant. given as

$$\text{Avg. power } P = \sum_{-\infty}^{\infty} |a_k|^2$$

$a_k \rightarrow$ F-series co-efficient.

Problem

(1) The fourier transform of a voltage signal is given as $V(\omega) = \frac{1}{1+j\omega}$ then the energy in the ^{correl.} time domain signal $v(t)$, is ?

Soln

$$V(\omega) = \frac{1}{1+j\omega}$$

$$v(t) = e^{-t} u(t)$$

$$E_{v(t)} = \int_{-\infty}^{\infty} |e^{-t} u(t)|^2 dt$$

$$= \int_0^{\infty} e^{-2t} dt$$

$$= \left[\frac{e^{-2t}}{-2} \right]_0^{\infty} = \frac{1}{2} \text{ Joules}$$

Energy in $v(\omega)$

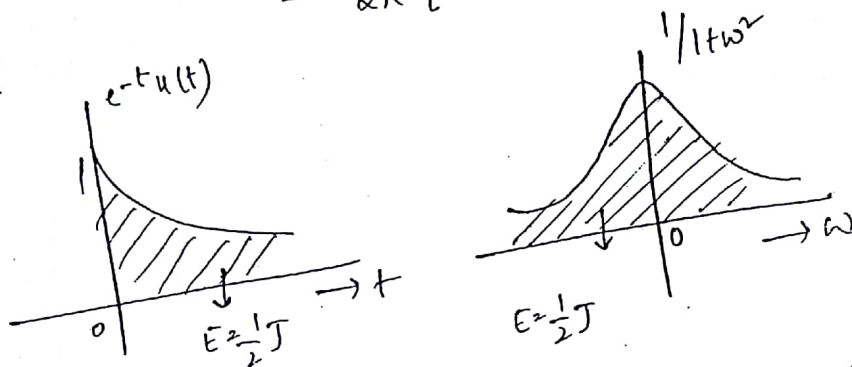
$$E_{v(\omega)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |v(\omega)|^2 d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1}{1+j\omega} \right)^2 d\omega$$

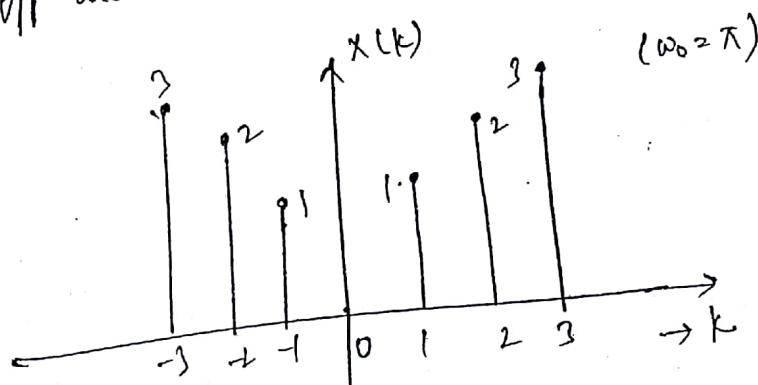
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(1+\omega^2)} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2\pi} (\tan^{-1}\omega)_{-\infty}^{\infty}$$

$$= \frac{1}{2\pi} \left[\frac{\pi}{2} + \frac{\pi}{2} \right] = \frac{1}{2} J.$$



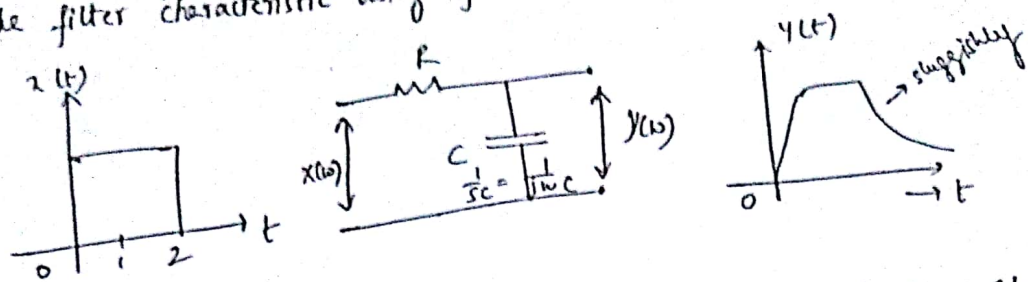
* State and prove Parseval's relation for the periodic signal whose F.S. coefficients are shown below.



9/2/13

Applications of Fourier Transforms:

(1) The filter characteristic using Fourier transforms:



$\Rightarrow$ The o/p response $y(t)$ is a distorted replica of the i/p response $x(t)$, due to the fact that a RC network act as a low pass filter. all the low freq. are easily accessed and high freq. are attenuated. Due to this the o/p response $y(t)$ rises sharply and falls sluggishly as shown above. But, in time domain it is not giving any of the characteristics like the frequencies that are attenuated, the band width of the system response etc. So, to analyse system response completely it has to be analysed in its Fourier transform as shown below

$$y(\omega) = x(\omega) = \frac{1}{j\omega C} \left(R + \frac{1}{j\omega C} \right)$$

$$\frac{y(\omega)}{x(\omega)} = H(\omega) = \frac{1}{1 + j\omega RC}$$

Let $R = 1 \Omega$, $C = 1 F$

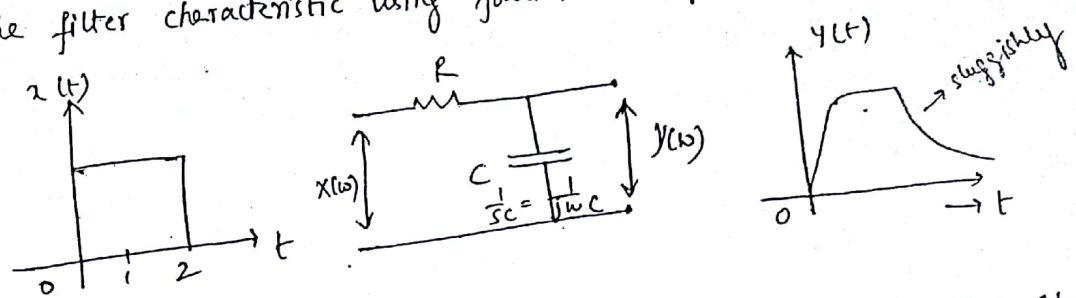
$$H(\omega) = \frac{1}{(1 + j\omega)}$$

$$|H(\omega)| = \frac{1}{\sqrt{1 + \omega^2}}$$

9/12/13

Applications of Fourier Transforms:

(1) The filter characteristic using Fourier transform:



$\Rightarrow$ The o/p response $y(t)$ is a distorted replica of the s/p response $x(t)$. Due to the fact that a RC network act as a low pass filter, all the low freq. are easily accessed and high freq. are attenuated. Due to this the o/p response $y(t)$ rises sharply and falls sluggishly as shown above.

But, in time domain it is not giving any of the characteristics like the frequencies that are attenuated, the band width of the system response etc. So, to analyse system response completely it has to be analysed in its Fourier transform as shown below

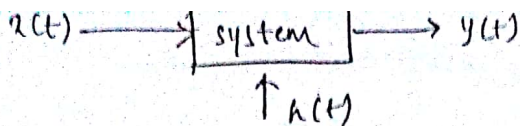
$$y(\omega) = x(\omega) \times \frac{1}{j\omega C} \left(R + \frac{1}{j\omega C} \right)$$

$$\frac{y(\omega)}{x(\omega)} = H(\omega) = \frac{1}{1 + j\omega RC}$$

$$\text{Let } R = 1 \Omega, C = 1 F$$

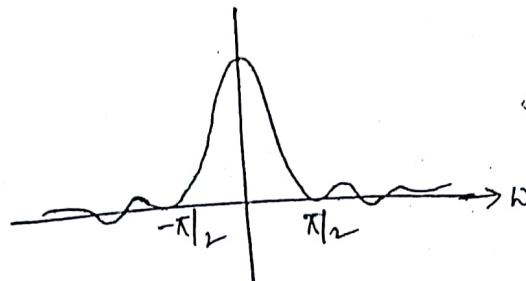
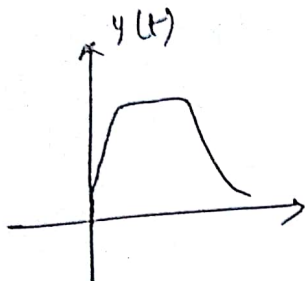
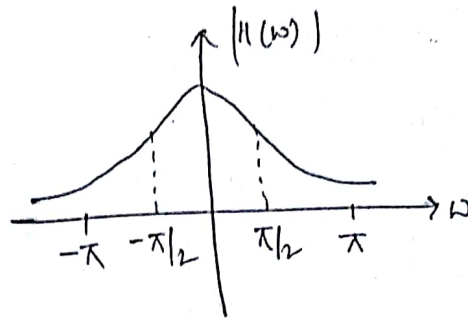
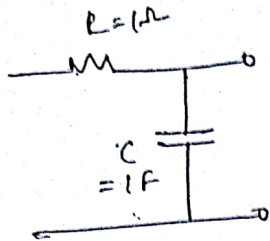
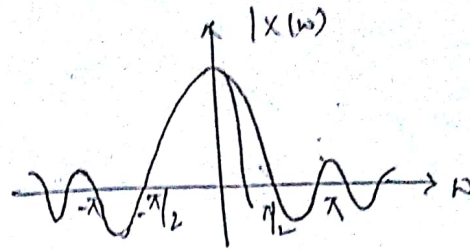
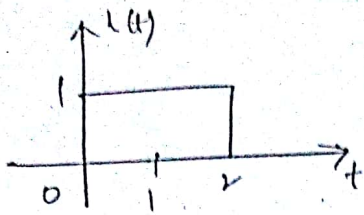
$$H(\omega) = \frac{1}{(1 + j\omega)}$$

$$|H(\omega)| = \underline{\underline{\frac{1}{\sqrt{1 + \omega^2}}}}$$



$$y(t) = x(t) * h(t)$$

$$Y(\omega) = X(\omega) H(\omega)$$



* Distortionless Transmission of a system response

→ For the system response to be distortionless the o/p response must be the exact replica (although there may be a change in its amplitude). To attain these characteristics let us consider

a "LTI" system

$$y(t) = kx(t-t_0)$$

F.T m.b.S

$$Y(\omega) = k \cdot e^{-j\omega t_0} X(\omega)$$

$$\frac{Y(\omega)}{X(\omega)} = H(\omega) = k e^{-j\omega t_0}$$

$$= r e^{j\theta}$$

3. * Bal

→ An

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** Sin

$$|H(\omega)| = k$$

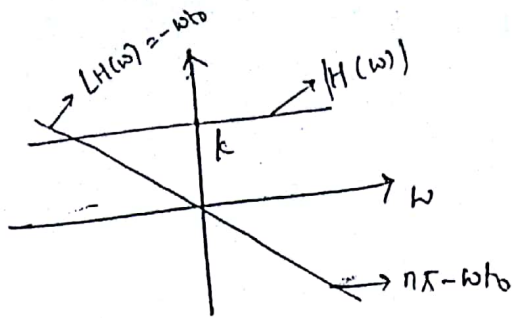
$$\angle H(\omega) = -\omega t_0$$

$\Rightarrow$

$$|H(\omega)| = k$$

$$\angle H(\omega) = n\pi - \omega t_0$$

$n \rightarrow \text{any integer}$



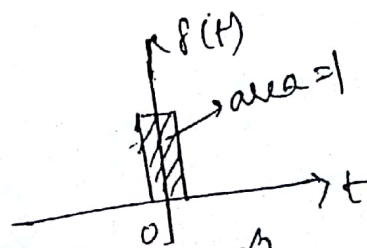
→ for the system to be distortionless a const. it has to have a constant magnitude response and linear phase response. So, if the system is distortionless it is absolutely a stable system.

3.* Band Width Requirements for distortionless transmission

→ An infinite band width is required for distortionless transmission but due to the physical limitations of the system response it is not possible. So, higher band width range must be used for distortionless transmission.

** Singularity functions The signals which does not satisfy the dirichlets conditions of F.T but having a finite energy or finite power are known as singularity functions.

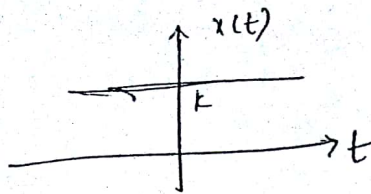
(1) unit impulse



$$X(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt$$

$$= 1$$

(2) constant k

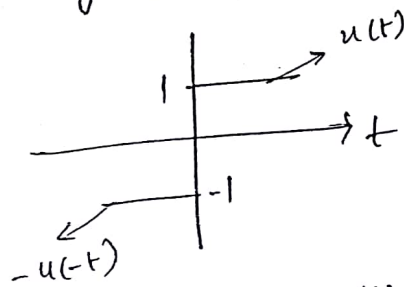


$$\delta(t) \leftrightarrow 1$$

$$1 \leftrightarrow 2\pi\delta(\omega) \rightarrow [\text{By duality theorem}]$$

$$\boxed{k \leftrightarrow 2\pi k\delta(\omega)}$$

(3) signum $\text{sgn}(t)$



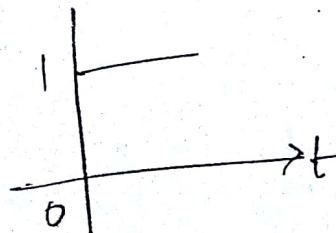
$$\text{sgn}(t) = u(t) - u(-t)$$

$$= \lim_{a \rightarrow 0} \int_{-\infty}^{\infty} e^{-at} u(t) - e^{at} u(-t) dt$$

$$\text{sgn}(t) \leftrightarrow \lim_{a \rightarrow 0} \left[\frac{1}{(a+j\omega)} - \frac{1}{(a-j\omega)} \right]$$

$$\boxed{\text{sgn}(t) \leftrightarrow \frac{2}{j\omega}}$$

(4) unit step $u(t)$



$$u(t) = \frac{1}{2} + \frac{1}{2} \text{sgn}(t)$$

$$u(t) \leftrightarrow \frac{1}{2} \times 2\pi\delta(\omega) + \frac{1}{2} \times \frac{2}{j\omega}$$

$$\boxed{u(t) \leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}}$$

* Fourier Transform of a periodic signal

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$\boxed{e^{jn\omega_0 t} \longleftrightarrow 2\pi \delta(\omega - n\omega_0)}$$

$$1 \longleftrightarrow 2\pi \delta(\omega)$$

$$1 \cdot e^{j\omega_0 t} \longleftrightarrow 2\pi \delta(\omega - \omega_0)$$

F.T on b-s

$$F(\omega) = \sum_{n=-\infty}^{\infty} c_n 2\pi \delta(\omega - n\omega_0)$$

$$\boxed{F(\omega) = 2\pi \sum_{n=-\infty}^{\infty} c_n \delta(\omega - n\omega_0)}$$

$$\text{where } \Rightarrow c_n = \frac{1}{T} \int_0^T f(t) \cdot e^{-jn\omega_0 t} \cdot dt$$

NOTE

→ The above F.T indicates that the spectrum of periodic signals is always discrete.

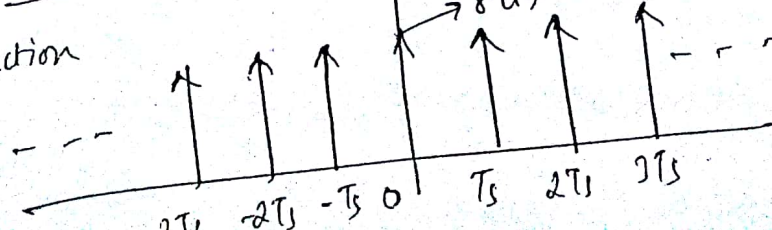
Problem

(1) find the F.T of periodic impulse train

Sol:

Periodic impulse train $\delta_{Ts}(t)$ / comb function

Ideal Function



$$c_n = \frac{1}{T_s} \int_0^{T_s} f(t) e^{-jn\omega_s t} dt$$

$$c_n = \frac{1}{T_s}$$

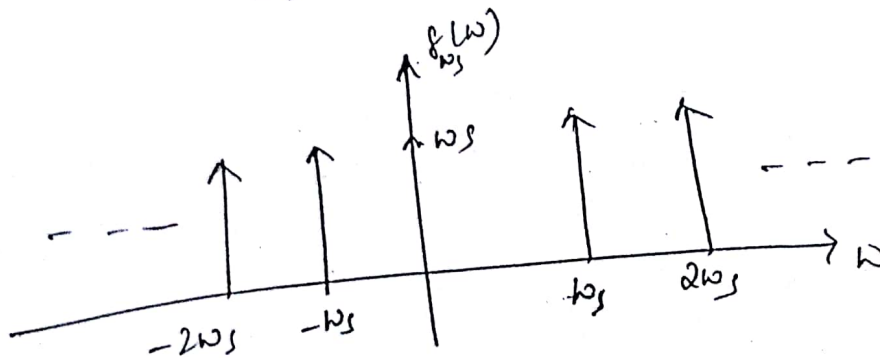
$$F(\omega) = 2\pi \sum_{n=-\infty}^{\infty} c_n \delta(\omega - n\omega_s)$$

$$= \frac{2\pi}{T_s} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$$

$$F(\omega) = \omega_s \delta \omega_s(\omega)$$

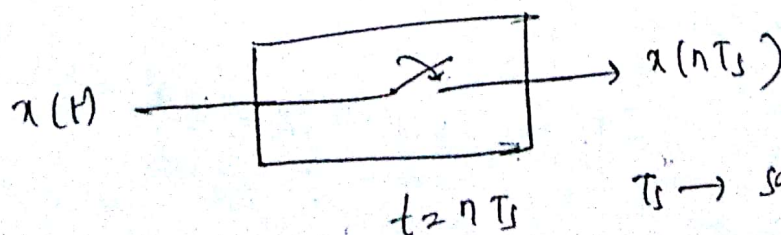
$$\delta \omega_s(\omega) = \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$$

↳ periodic impulse train in freq.



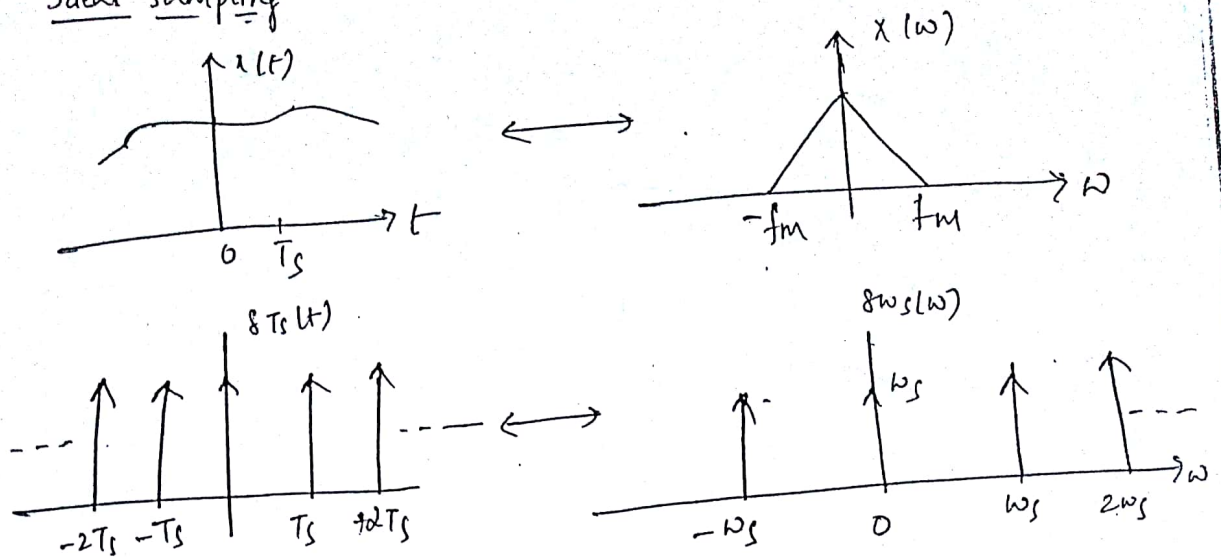
* Sampling Theorem

→ Sampling: The process of converting a continuous time signal into a discrete time signal which is uniformly spaced

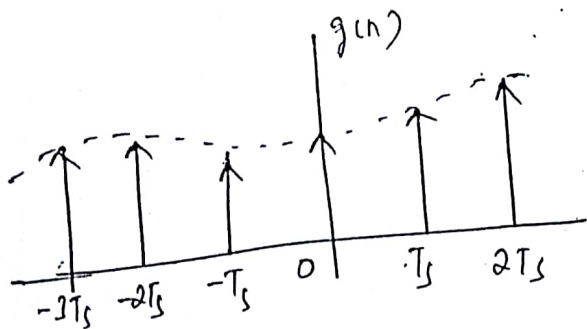


$T_s \rightarrow$ sampling time
 $n \rightarrow$ any integer
 $0, \pm 1, \pm 2, \dots$

Ideal sampling



$$g(n) = x(nT_s) = x(t) \times \delta_{T_s}(t)$$



$$\begin{aligned} g(n) &= x(t) \times \delta_{T_s}(t) \\ &= x(t) \times \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \end{aligned}$$

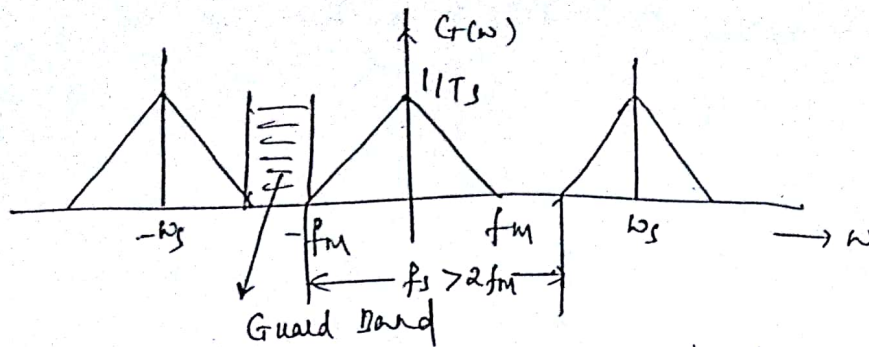
F.T on b.s

$$G(\omega) = \frac{1}{2\pi} \left[x(\omega) * \omega_s \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s) \right]$$

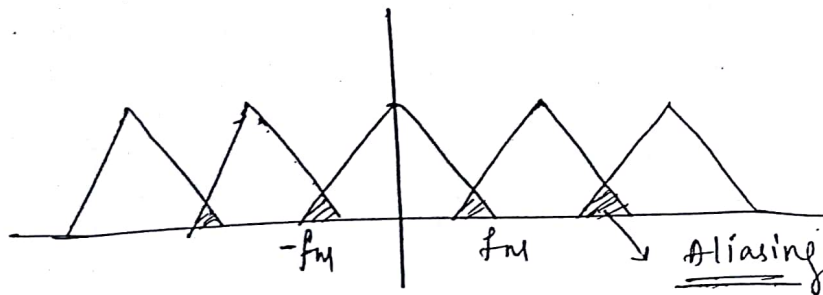
$$= \frac{\omega_s}{2\pi} \sum_{n=-\infty}^{\infty} x(\omega) * \delta(\omega - n\omega_s)$$

$$G(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

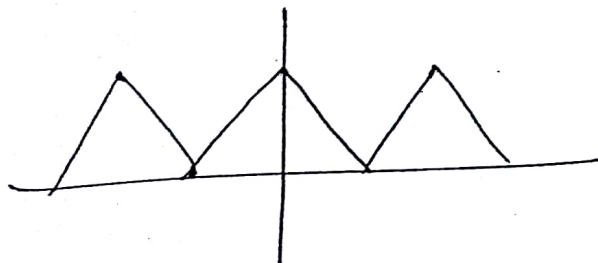
Case 1: $f_s > 2f_m$ (Nyquist rate)



Case 2: $f_s < 2f_m$



Case 3: $f_s = 2f_m$ → Ideal sampling
→ Nyquist Rate



→ Sampling rate or sampling frequency or Nyquist rate (f_s):

⇒ The rate at which the sampling is done without distortion is known as sampling rate

$$f_s \geq 2f_m$$

f_m → max. freq. component in signal

→ Sampling time or Sampling Interval / Nyquist interval

⇒ The minimum time required for sampling a signal without aliasing.

$$T_s = \frac{1}{f_s}$$

$$T_s \leq \frac{1}{2f_m}$$

→ Guard Band: The band which protects from high freq. components without getting mix from low frequency components.

$$\begin{matrix} f_s > 2f_m \\ T_s < \frac{1}{2f_m} \end{matrix}$$

Due to this guard band the complete reconstruction of the original signal is only applicable

→ Aliasing: The phenomena in which the high frequencies taking the identity of low frequency components is known as aliasing. Due to this the complete reconstruction of the original signal is not applicable

$$\begin{matrix} f_s < 2f_m \\ T_s > \frac{1}{2f} \end{matrix}$$

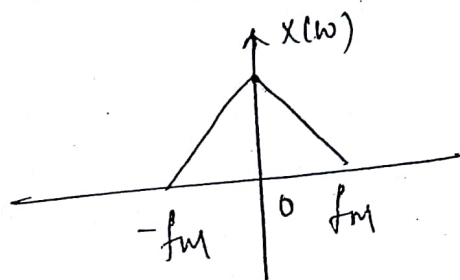
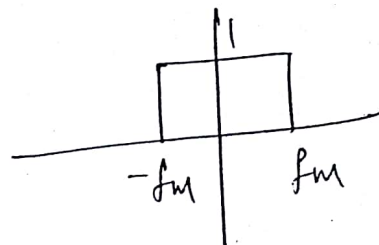
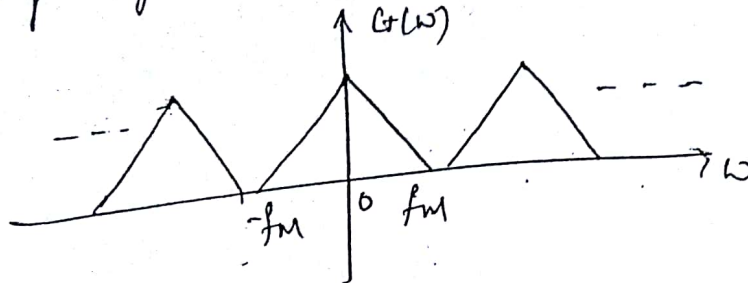
→ Ideal sampling: If the sampling rate is exactly equal to max. freq. component of the signal.

$$\begin{matrix} f_s = 2f_m \\ T_s = \frac{1}{2f_m} \end{matrix}$$

(f_s):
without

* Reconstruction of original signal Sampling Theorem

⇒ Any band limited signal which has no frequency components higher than f_m Hz can be easily recovered from the knowledge of its samples taken at a rate slightly higher than twice the max. freq component.



$$X(\omega) = G(\omega) * H(\omega)$$

$$H(\omega) = 1; -\omega_m < \omega < \omega_m$$

↳ Ideal LFF Transfer function

* State and Prove the reconstruction of a sampling using mathematical analysis.

$$G_s(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

$$T = \frac{1}{2f_m}$$

$$\omega_s = \frac{2\pi}{T} = 4\pi f_m \Rightarrow \omega_s = 2\omega_m$$

$$G_s(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\omega - 2n\omega_s)$$

$G_s(\omega)$ is obtained by passing through gate function
 $\downarrow$
 $H(\omega) 2\omega_m$

$$G_s(\omega) H_{2\omega_m}(\omega) = \frac{1}{T} X(\omega)$$

$$X(\omega) = T G_s(\omega) H_{2\omega_m}(\omega)$$

Apply conv. theorem in time domain

$$x(t) = T g_s(t) * \frac{\omega_m}{\pi} \text{sinc}(\omega_m t)$$

$$g_s(t) = \sum_{n=-\infty}^{\infty} g_n \delta(t - nT_s)$$

$$x(t) = \sum_{n=-\infty}^{\infty} g_n \delta(t - nT_s) * \text{sinc}(\omega_m t)$$

By properties of impulse function

$$x(t) = \sum_{n=-\infty}^{\infty} g_n \text{sinc}(\omega_m (t - nT))$$

(1) For the given continuous time signal $x(t) = 2\cos(250\pi t)$. Find

(i) f_m (ii) f_s (iii) T_s (iv) Discrete signal $x(n)$ obtained at ideal sampling (v) Discrete signal obtained at sampling time of 2ms.

Sol:

$$x(t) = 2\cos(250\pi t)$$

$$\omega_m = 250\pi$$

$$2\pi f_m = 250\pi$$

$$(1) \boxed{f_m = 125 \text{ Hz}}$$

$$(2) f_s \geq 2f_m$$
$$2 \times 125 \text{ Hz}$$

$$\boxed{f_s \geq 250 \text{ Hz}}$$

$$(3) T_s \leq \frac{1}{2f_m}$$
$$\leq \frac{1}{250 \text{ Hz}}$$

$$\boxed{T_s \leq 4 \text{ ms}}$$

$$(4) T_s = \frac{1}{2f_m} = 4 \text{ msec}$$

$$x(nT_s) = 2\cos(250\pi(nT_s))$$

$$= 2\cos(250\pi(n \times \frac{1}{250}))$$

$$\boxed{x(nT_s) = 2\cos(\pi n)}$$

$$(5) x(nT_s) = 2\cos(250\pi n \times \frac{T_s}{1000})$$

$T_s = 2 \text{ msec}$
 $\frac{2}{1000} = \frac{1}{500}$

$$\boxed{x(n) = 2\cos(\frac{\pi}{2}n)}$$

Find
at
mg

- (2) Find for the continuous time signal $x(t) = 2\sin(350\pi t) + 5\cos(125\pi t) + 3\sin(500\pi t)$.
Find the f_m , nyquist rate (f_s), T_s , discrete signal $x(n)$ obtained at sampling time of 1ms

Sol:- $x(t) = 2\sin(350\pi t) + 5\cos(125\pi t) + 3\sin(500\pi t)$

(1) ~~$f_m = 500\pi$~~
 $f_m = 250$
 $f_m = 250 \text{ Hz}$

(2) $f_s \geq 2f_m$
 $= 2 \times 250$
 $= 500 \text{ Hz}$

(3) $T_s \leq \frac{1}{2f_m}$
 $\leq \frac{1}{2 \times 250}$
 $\leq 2 \text{ ms}$

(4) $x(nT_s) = 2\sin(350\pi nT_s) + 5\cos(125\pi(nT_s)) + 3\sin(500\pi(nT_s))$

$= 2\sin\left(350\pi n \times \frac{1}{1000}\right) + 5\cos\left(125\pi\left(n \times \frac{1}{1000}\right)\right) + 3\sin\left(500\pi\left(n \times \frac{1}{1000}\right)\right)$

$= 2\sin\left(\frac{7n\pi}{20}\right) + 5\cos\left(\frac{n\pi}{8}\right) + 3\sin\left(\frac{n\pi}{2}\right)$

- ***
(3) A continuous time signal $x(t) = 150\cos(150\pi t - 60^\circ)$ is sampled at a freq of ~~2~~ 150 Hz. Find the fundamental period of discrete signal $x(n)$ obtained after sampling.

Sol:- $x(t) = 3\cos(150\pi t - 60^\circ)$

$f_s = 350 \text{ Hz}$

$T_s = \frac{1}{f_s} = \frac{1}{350}$

$t \rightarrow nT_s$

$x(nT_s) = 3\cos\left(150\pi \times n \times \frac{1}{350} - 60^\circ\right)$

$= 3\cos\left(\frac{3\pi n}{7} - \frac{\pi}{3}\right)$

$\omega = n\left(\frac{2\pi}{N}\right)$

$\frac{3\pi}{7} = 3\left(\frac{2\pi}{N}\right)$

$N = 14$

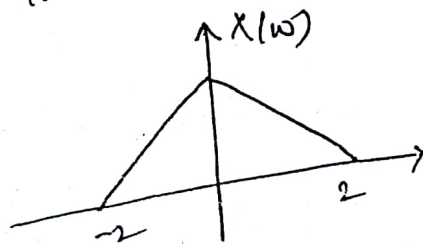
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* Sketch the spectrum of the sampled signal

Case (i): At $t = \frac{\pi}{3} \text{ sec}$

(ii): $t = \pi/2 \text{ sec}$

(iii): $T = \frac{2\pi}{3}$



Also find in which of the following cases Nyquist rate is satisfying the condition of aliasing, ideal sampling and width of guard band if applicable

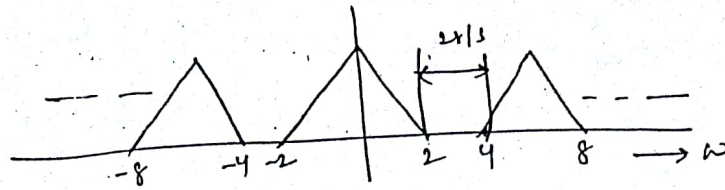
Sol:- (i) $T = \pi/2 \text{ s}$

$\omega = \frac{2\pi}{T} = \frac{2\pi}{\pi/2} = 4 \text{ r/s}$

(1) N.R ✓

(2) width of G.B is

2 r/s

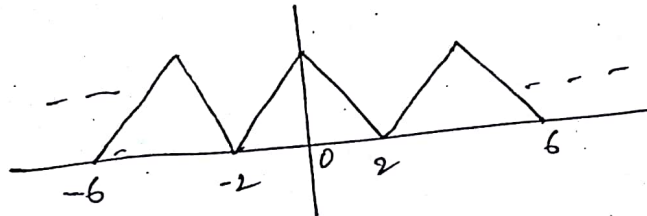


(2) $T = \frac{\pi}{2} s$

$\omega = \frac{2\pi}{\pi/2} = 4 \text{ r/s}$

(1) N.R ✓

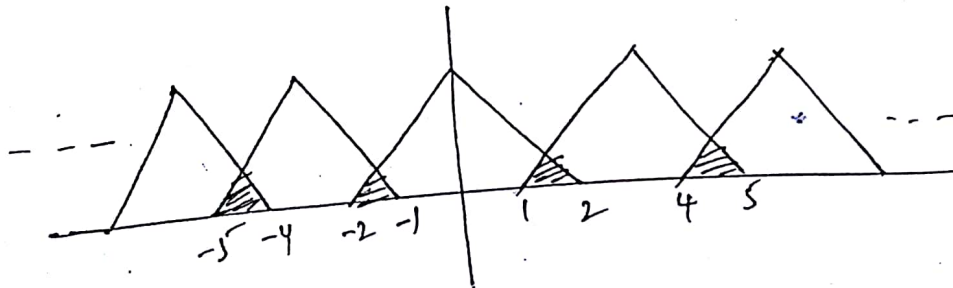
(2) ideal sampling



(3) $T = \frac{2\pi}{3} \text{ r/s}$

$\omega = \frac{2\pi}{(2\pi/3)} = 3 \text{ r/s}$

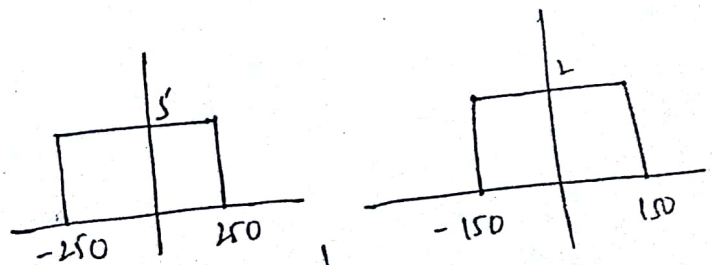
(1) N.R is not satisfied
(2) case of aliasing



* find the nyquist interval of the signal $x(t) = 2\text{sinc}(300t) + 5\sin(500t)$

Sol: $x(t) = 2\text{sinc}(300t) + 5\sin(500t)$

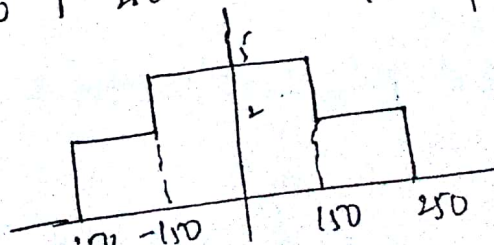
=



$f_s \geq 2f_m$

$f_s \geq 2(250) \text{ Hz}$

$T_s \geq 500 \text{ Hz} \Rightarrow T_s \leq \frac{1}{\text{ms}}$



* In spite of no. of applications to F.T. F.T. can be analysed only for bounded signals and for stable systems. In order to analyse both stable and unstable systems, bounded and unbounded signals Laplace transforms are used.

* Laplace Transforms

$$\mathcal{L}[x(t)] = X(s)$$

$$\boxed{X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt} \Rightarrow \text{L.T}$$

$$\text{Let } s = \sigma + j\omega$$

$$\begin{aligned} X(\sigma + j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt \\ &= \int_{-\infty}^{\infty} x(t) e^{-\sigma t} \cdot e^{-j\omega t} dt \end{aligned}$$

$$\text{If } \sigma = 0$$

$$\boxed{X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt} \rightarrow \text{F.T}$$

$$\boxed{s = j\omega} \rightarrow \text{Relation b/w L.T \& F.T}$$

Inverse d.T:

$$\mathcal{L}^{-1}[X(s)] = x(t)$$

$$\boxed{x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds}$$

1) Find the L.T of the signal $e^{-at}u(t)$ and also find its region of convergence (R.O.S)

Sol: $x(t) = e^{-at}u(t)$

$$X(s) = \int_{-\infty}^{\infty} e^{-at}u(t) e^{-st} dt$$

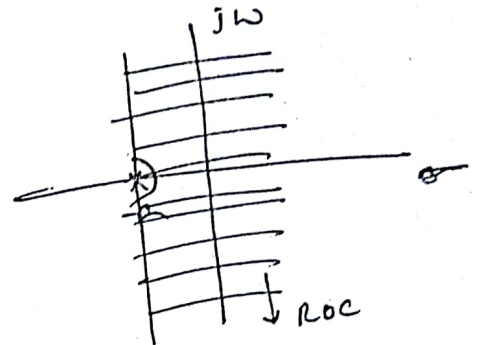
$$= \int_0^{\infty} e^{-(s+a)t} dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_0^{\infty}$$

$$\Rightarrow X(s) = \frac{1}{s+a}$$

$$(s+a) > 0$$

$$\boxed{\text{Re}(s) > -a} \rightarrow \text{ROC}$$



(2) $x(t) = -e^{-at}u(-t)$

$$X(s) = \int_{-\infty}^{\infty} -e^{-at}u(-t) e^{-st} dt$$

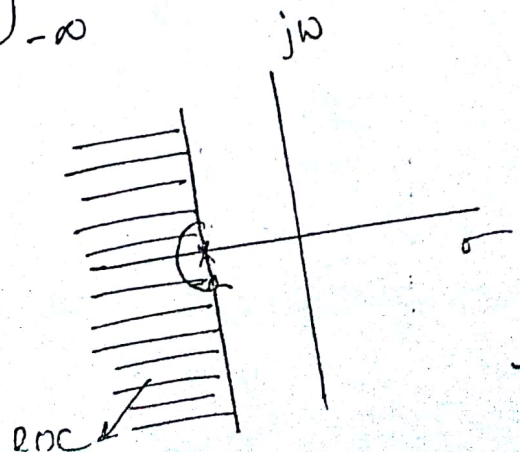
$$= - \int_{-\infty}^0 e^{-(a+s)t} dt$$

$$= - \int_{-\infty}^0 e^{-(a+s)t} dt$$

$$= - \left[\frac{e^{-(s+a)t}}{s+a} \right]_{-\infty}^0$$

$$(s+a) < 0$$

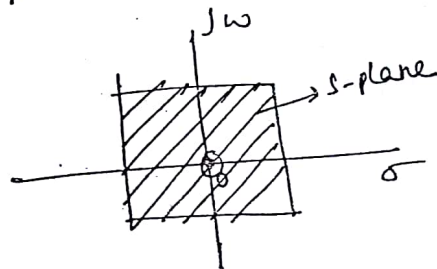
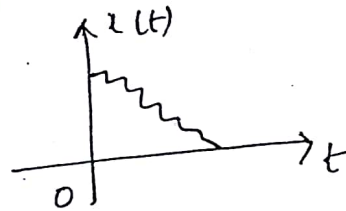
$$\Rightarrow \text{Re}(s) < -a$$



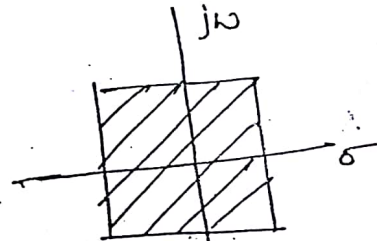
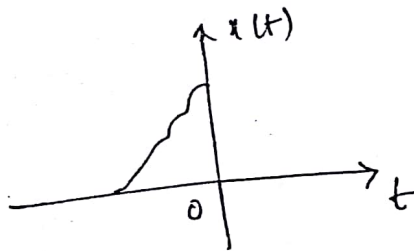
* Region of Convergence: The range of values of 's' for which the signal $x(t)$ converges is known as region of Convergence.

Properties of ROC:

- (1) The ROC should not contain any poles. and ROC doesn't depend on zeros. So, they can be anywhere in the ROC
- (2) If the signal $x(t)$ is finite and right-sided ROC is the region of entire s-plane except at ' $s=0$ '.

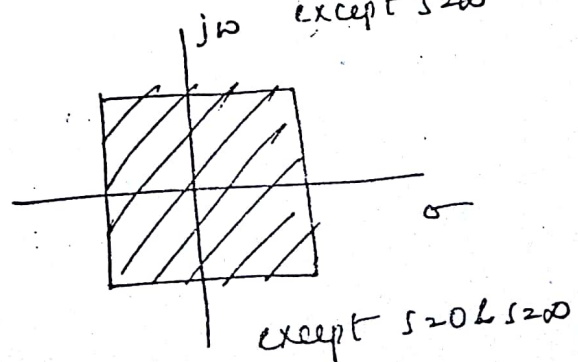
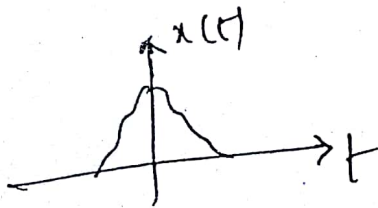


(3)



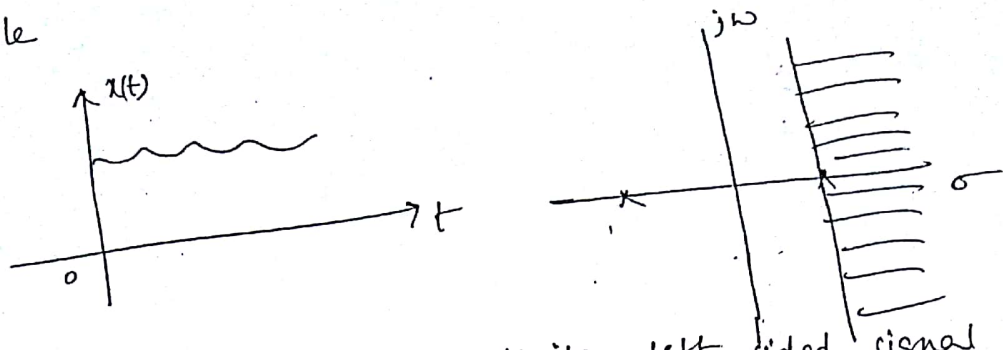
If $x(t)$ left sided then ROC is entire except $s=\infty$.

(4)

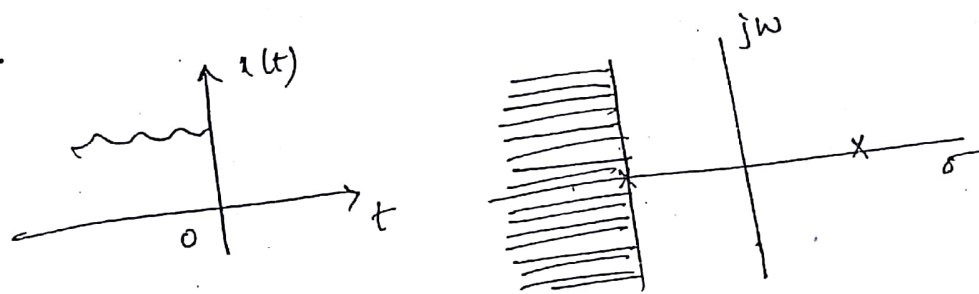


If the signal $x(t)$ is two-sided, ROC is the region of entire s-plane except $s=0$ & $s=\infty$.

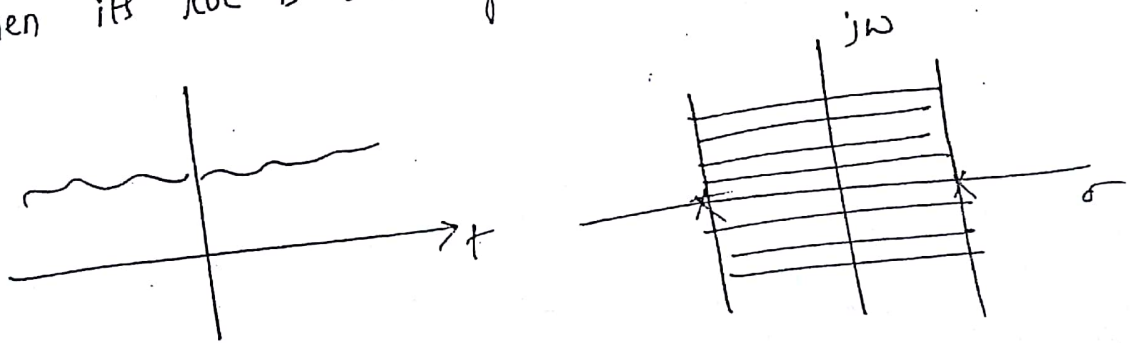
(5) If the signal $x(t)$ is an infinite right sided signal then its ROC is the region to the right of the rightmost pole



(6) If the signal $x(t)$ is an infinite left sided signal then its ROC is the region to the left of the leftmost pole



(7) If the signal $x(t)$ is an infinite two sided signal then its ROC is the region common b/w the poles.



Problem

Find L.T of the following signals and indicate its ROC

(1) $x(t) = e^{-t}u(t) + e^{-3t}u(t)$

Soln

$$X(s) = \int_{-\infty}^{\infty} e^{-t}u(t) + e^{-3t}u(t) e^{-st} dt$$

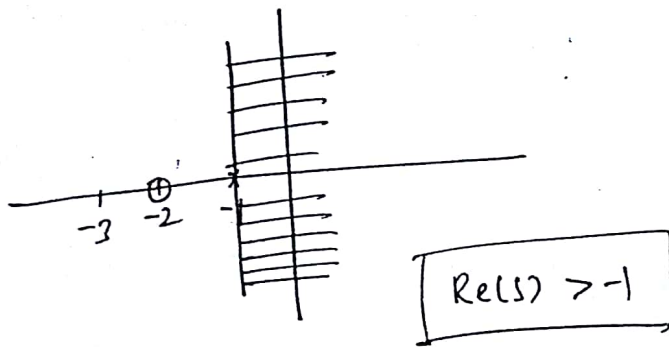
$$= \int_0^{\infty} e^{-t} dt \cdot e^{-st} + \int_0^{\infty} e^{-3t} dt \cdot e^{-st}$$

$$= \int_0^{\infty} e^{-t} e^{-st} dt + \int_0^{\infty} e^{-2t} e^{-st} dt$$

$$= \int_0^{\infty} e^{-(s+1)t} dt + \int_0^{\infty} e^{-t(s+2)} dt$$

$$= \frac{1}{s+1} + \frac{1}{s+2}$$

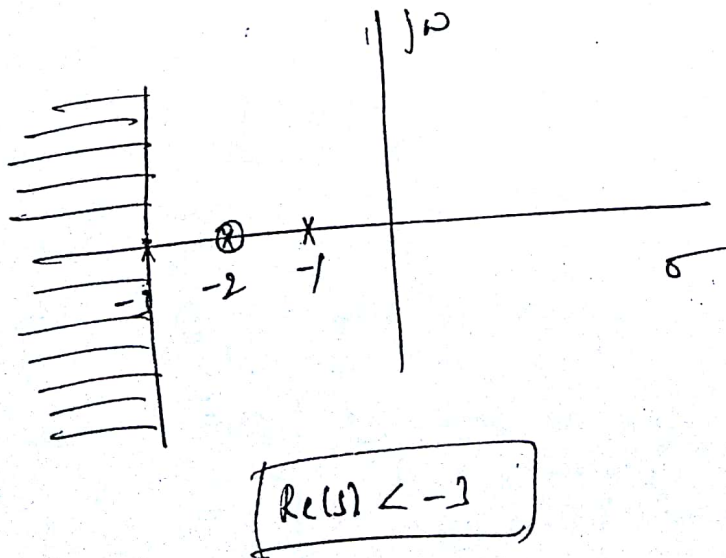
$$= \frac{2(s+2)}{(s+1)(s+2)}$$



(2)

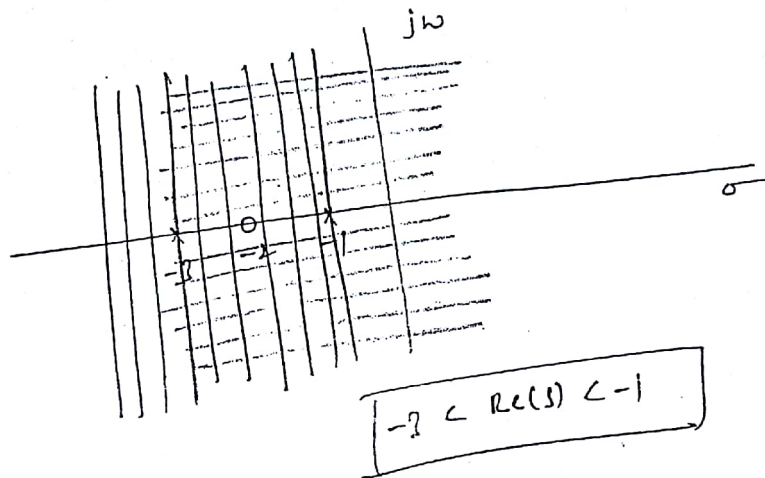
$$x(t) = -e^{-t} u(-t) - e^{-2t} u(-t)$$

$$X(s) = \frac{2(s+2)}{(s+1)(s+2)}$$



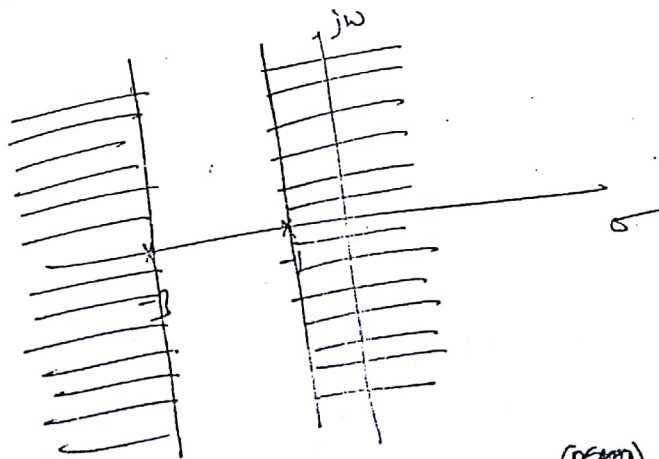
(3) $x(t) = -e^{-t}u(-t) + e^{-3t}u(t)$

$$X(s) = \frac{2(s+2)}{(s+1)(s+3)}$$



(4) $x(t) = e^{-t}u(t) - e^{-3t}u(-t)$

$$X(s) = \frac{2(s+2)}{(s+1)(s+3)}$$



Since, no common ~~ROC~~ ROC b/w the poles
 so, L.T doesn't exist.

And the L.T of the signal $x(t) = e^{-5t} + e^{-2t}u(t)$ and also find its ROC

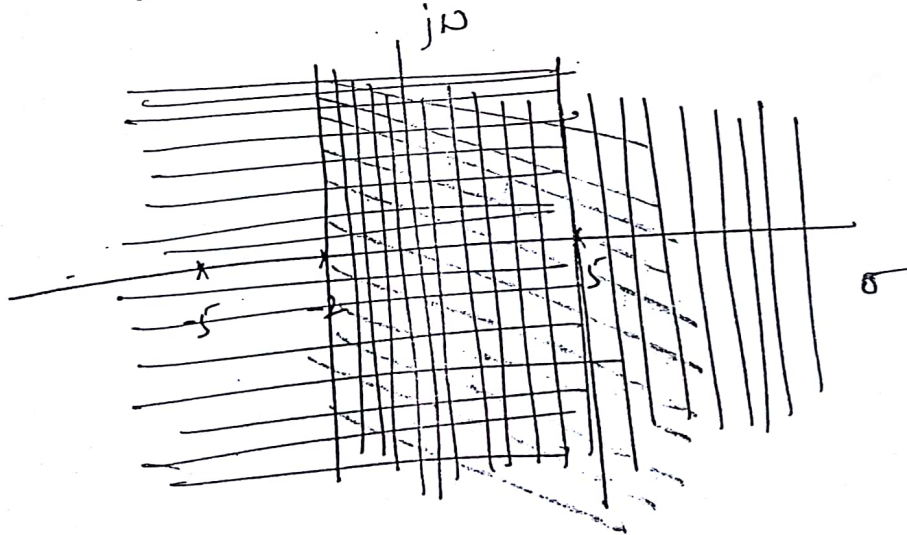
Sol:- $x(t) = e^{-5|t|} + e^{-2t}u(t)$

$$X(s) = \int_{-\infty}^{\infty} e^{-5|t|} e^{-st} dt + \int_{-\infty}^{\infty} e^{-2t} e^{-st} u(t) dt$$

$$= \int_{-\infty}^0 e^{5t} e^{-st} dt + \int_0^{\infty} e^{-5t} e^{-st} dt + \frac{1}{s+2}$$

$$= \int_{-\infty}^0 e^{(5-s)t} dt + \int_0^{\infty} e^{-(5+s)t} dt + \frac{1}{s+2}$$

$$= e^{-5t}u(t) + e^{5t}u(-t) + e^{-2t}u(t) = \frac{1}{s+5} - \frac{1}{s-5} + \frac{1}{s+2}$$



For the given signals find the L.T and check their

ROC

(1) $\delta(t)$

(2) $u(t)$

(3) $-u(-t)$

(4) $t u(t)$

(5) $-t u(-t)$

(6) $t e^{-at} u(t)$

(7) $-t e^{-at} u(-t)$

(8) $\cos bt u(t)$

(9) $\sin bt u(t)$

(10) $e^{-at} \cos bt u(t)$

$$(11) e^{-at} \sin bt u(t)$$

$$(15) e^{-at} \sinh bt u(t)$$

$$(12) \cosh bt u(t)$$

$$(13) \sinh bt u(t)$$

$$(14) e^{-at} \cosh bt u(t)$$

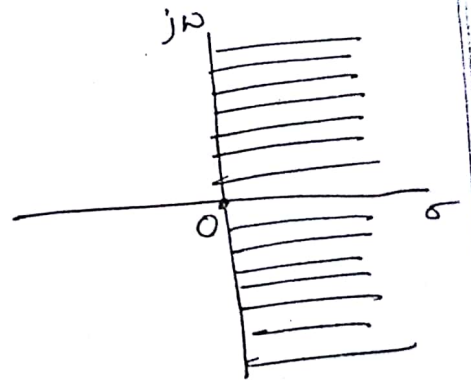
Soln

$$(1) \delta(t)$$

$$\begin{aligned} \mathcal{L}\{\delta(t)\} &= \int_{-\infty}^{\infty} \delta(t) e^{-st} dt \\ &= 1 \end{aligned}$$

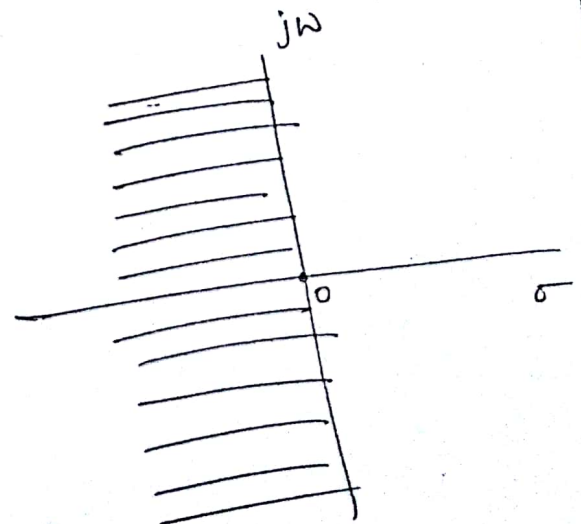
$$(2) u(t)$$

$$\begin{aligned} \mathcal{L}\{u(t)\} &= \int_{-\infty}^{\infty} u(t) e^{-st} dt \\ &= \int_0^{\infty} e^{-st} dt = \left[\frac{e^{-s}}{-s} \right]_0^{\infty} \\ &= 1/s \end{aligned}$$



$$(3) -u(-t)$$

$$\begin{aligned} \mathcal{L}\{-u(-t)\} &= \int_{-\infty}^{\infty} -u(-t) e^{-st} dt \\ &= \int_{-\infty}^0 -e^{-st} dt \\ &= - \left[\frac{e^{-s}}{-s} \right]_{-\infty}^0 \\ &= 1/s \end{aligned}$$



(4)

$t u(t)$

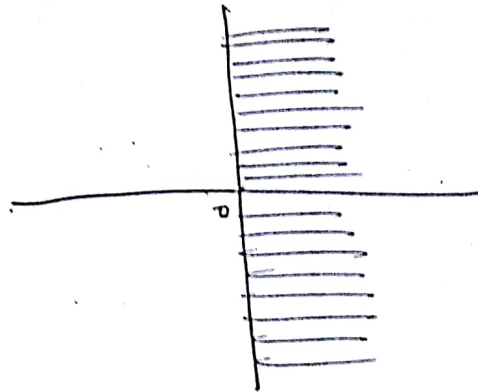
$$\mathcal{L}\{T\} = \int_{-\infty}^{\infty} t \cdot u(t) e^{-st} dt$$

$$= \int_0^{\infty} t \cdot e^{-st} dt$$

$$= e^{-st} \left(-\frac{t}{s} + \frac{1}{s^2} \right) \Big|_0^{\infty}$$

$$= 1/s^2$$

$$\begin{aligned} & t \cdot \frac{e^{-st}}{-s} - \int 1 \cdot \frac{e^{-st}}{-s} \\ & -t \cdot \frac{e^{-st}}{s} + \frac{e^{-st}}{s^2} \Big|_0^{\infty} \\ & e^{-st} \left(-\frac{t}{s} + \frac{1}{s^2} \right) \end{aligned}$$



(5)

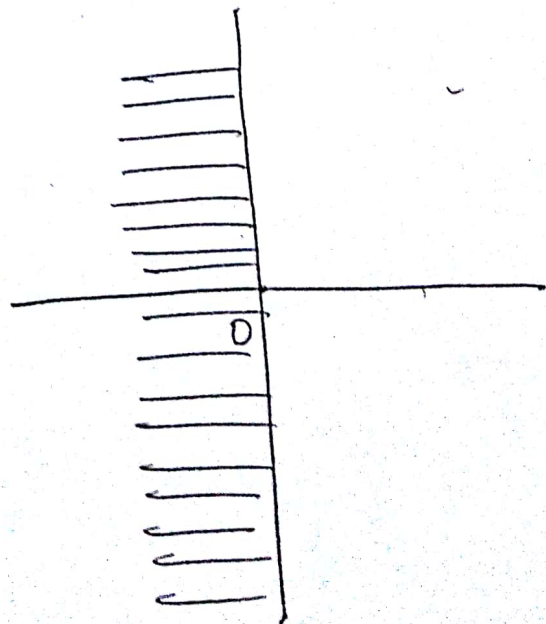
$-t u(-t)$

$$\mathcal{L}\{T\} = - \int_{-\infty}^{\infty} t u(-t) e^{-st} dt$$

$$= - \int_{-\infty}^0 t e^{-st} dt$$

$$= - \left[e^{-st} \left(-\frac{t}{s} + \frac{1}{s^2} \right) \right]_{-\infty}^0$$

$$= 1/s^2$$

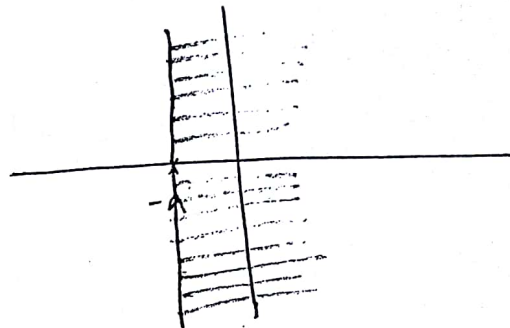


6. $t e^{-at} u(t)$

$$\mathcal{L}\{T\} = \int_{-\infty}^{\infty} t \cdot e^{-at} u(t) e^{-st} dt$$

$$= \int_0^{\infty} t \cdot e^{-t(a+s)} dt$$

$$= \frac{1}{(s+a)^2}$$

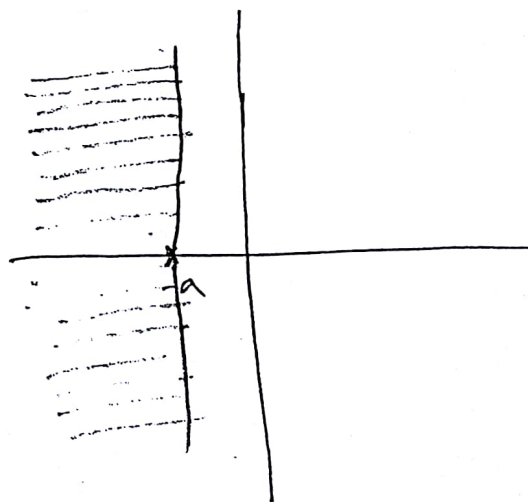


7. $-t e^{-at} u(-t)$

$$\mathcal{L}\{T\} = - \int_{-\infty}^{\infty} t e^{-at} u(-t) e^{-st} dt$$

$$= - \int_{-\infty}^0 t e^{-t(s+a)} dt$$

$$= \frac{1}{(s+a)^2}$$



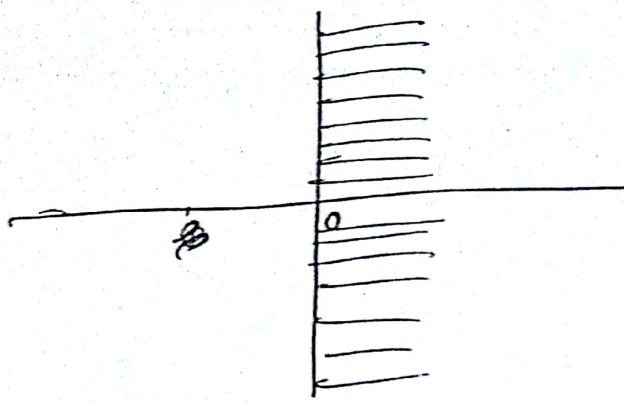
8. $\cos bt u(t)$

$$\mathcal{L}\{T\} = \int_{-\infty}^{\infty} \cos bt u(t) e^{-st} dt$$

$$= \int_0^{\infty} \cos bt e^{-st} dt$$

$$= \frac{e^{-st}}{s^2 + b^2} (s \cos bt + b \sin bt) \Big|_0^{\infty}$$

$$= \frac{1}{s^2 + b^2} (s) = \frac{s}{s^2 + b^2}$$



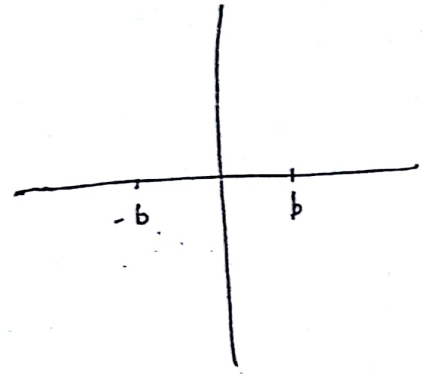
9. $\sin bt u(t)$

$$L.T = \int_{-\infty}^{\infty} \sin bt u(t) \cdot e^{-st} dt$$

$$= \int_0^{\infty} \sin bt e^{-st} dt$$

$$= \left[\frac{e^{-st}}{s^2 + b^2} (s \sin bt + b \cos bt) \right]_0^{\infty}$$

$$= \underline{\underline{\frac{b}{s^2 + b^2}}}$$

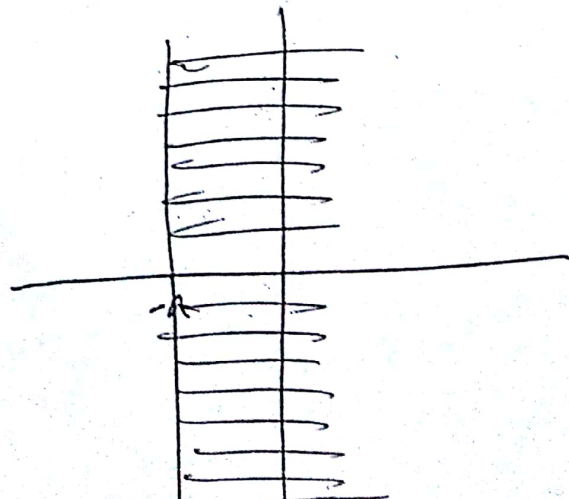


10. $e^{-at} \cos bt u(t)$

$$L.T = \int_{-\infty}^{\infty} e^{-at} \cos bt u(t) e^{-st} dt$$

$$= \int_0^{\infty} \cos bt e^{-t(s+a)} dt$$

$$= \frac{s+a}{(s+a)^2 + b^2}$$

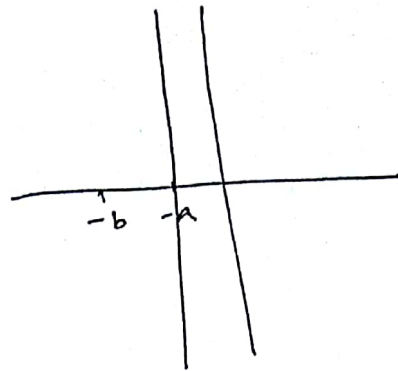


11. $e^{-at} \sin bt \, u(t)$

$$\mathcal{L}\{T\} = \int_{-\infty}^{\infty} e^{-at} \sin bt \, u(t) e^{-st} dt$$

$$= \int_0^{\infty} \sin bt \, e^{-t(s+a)} dt.$$

$$= \frac{b}{(s+a)^2 + b^2}$$



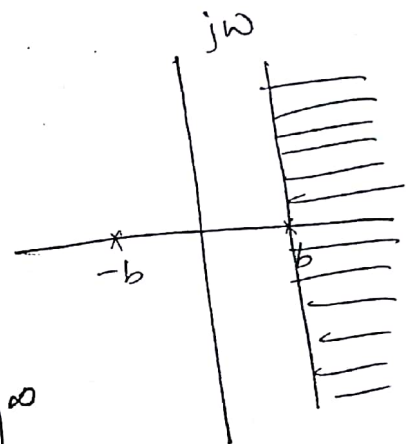
12. $\cosh bt \, u(t)$

$$\mathcal{L}\{T\} = \int_{-\infty}^{\infty} \cosh bt \, u(t) \cdot e^{-st} dt.$$

$$= \int_0^{\infty} \cosh bt \, e^{-st} dt$$

$$= \frac{e^{-st}}{s^2 - b^2} \left[s \cosh bt - b \sinh bt \right]_0^{\infty}$$

$$= \frac{s}{s^2 - b^2}$$

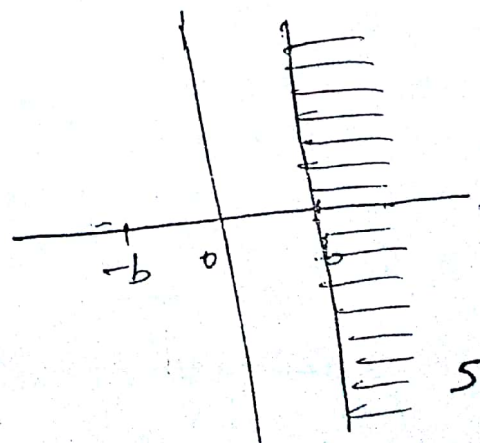


13. $\sinh bt \, u(t)$

$$\mathcal{L}\{T\} = \int_{-\infty}^{\infty} \sinh bt \, u(t) e^{-st} dt$$

$$= \int_0^{\infty} \sinh bt \, e^{-st} dt$$

$$= \frac{b}{s^2 - b^2}$$



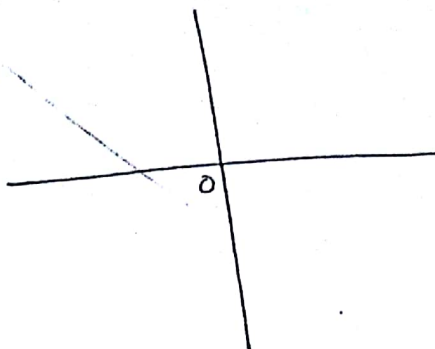
14

$$e^{-at} \cosh bt \, u(t)$$

$$L.T = \int_{-\infty}^{\infty} e^{-at} \cosh bt \, u(t) e^{-st} dt$$

$$= \int_0^{\infty} e^{-t(s+a)} \cosh bt \, dt$$

$$= \frac{s+a}{(s+a)^2 - b^2}$$



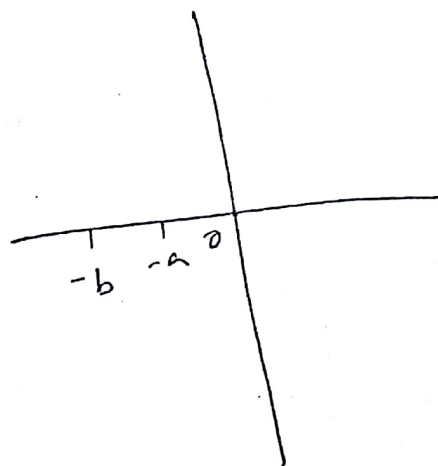
15.

$$e^{-at} \sinh bt \, u(t)$$

$$L.T = \int_{-\infty}^{\infty} e^{-at} \sinh bt \, u(t) e^{-st} dt$$

$$= \int_0^{\infty} e^{-t(s+a)} \sinh bt \, dt$$

$$= \frac{b}{(s+a)^2 - b^2}$$



13/12/19

S.NO	signal	A.T	ROC
(1)	$\delta(t)$	1	Entire s-plane
(2)	$u(t)$	$1/s$	$\text{Re}(s) > 0$
(3)	$t \cdot u(t)$	$1/s^2$	$\text{Re}(s) > 0$
(4)	$t^n u(t)$	$\frac{n!}{s^{n+1}}$	$\text{Re}(s) > 0$
(5)	$e^{-at} u(t)$	$\frac{1}{s+a}$	$\text{Re}(s) > -a$
(6)	$t \cdot e^{-at} u(t)$	$\frac{1}{(s+a)^2}$	$\text{Re}(s) > -a$
(7)	$t^n e^{-at} u(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\text{Re}(s) > -a$
(8)	$- (u(-t))$	$1/s$	$\text{Re}(s) < 0$
(9)	$-t u(-t)$	$1/s^2$	$\text{Re}(s) < 0$
(10)	$-t^n u(-t)$	$\frac{n!}{s^{n+1}}$	$\text{Re}(s) < 0$
(11)	$-e^{-at} u(-t)$	$1/s+a$	$\text{Re}(s) < -a$
(12)	$-te^{-at} u(-t)$	$1/(s+a)^2$	$\text{Re}(s) < -a$
(13)	$-t^n e^{-at} u(-t)$	$\frac{n!}{(s+a)^{n+1}}$	$\text{Re}(s) < -a$
(14)	$\cos bt u(t)$	$\frac{s}{s^2+b^2}$	$\text{Re}(s) > 0$
(15)	$\sin bt u(t)$	$\frac{b}{s^2+b^2}$	$\text{Re}(s) > 0$
(16)	$e^{-at} \cos bt u(t)$	$\frac{s+a}{(s+a)^2+b^2}$	$\text{Re}(s) > -a$

* Properties:

<u>S.NO</u>	<u>Name of property</u>
1.	Linearity
2.	Time scaling
3.	Time shifting
4.	Shifting in s-domain
5.	Diff in time
6.	Diff in s-domain
7.	Integration in time domain
8.	Integration in s-domain
9.	convolution in time

Property

$$a_1x_1(t) + b_2x_2(t) \xleftrightarrow{\mathcal{L}T} a_1X_1(s) + b_2X_2(s)$$

$$x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{s}{a}\right)$$

$$x(t-t_0) \longleftrightarrow e^{-st_0} X(s)$$

$$e^{s_0 t} x(t) \longleftrightarrow X(s-s_0)$$

$$\begin{aligned} \frac{d}{dt} x(t) &\longleftrightarrow sX(s) \\ &\longleftrightarrow [sX(s) - x(0^+)] \end{aligned}$$

$$\frac{d^n}{dt^n} x(t) \longleftrightarrow s^n X(s)$$

$$t x(t) \longleftrightarrow \frac{d}{ds} X(s)$$

$$\begin{aligned} \int_{-\infty}^t x(t) dt &\longleftrightarrow \frac{X(s)}{s} \\ &\longleftrightarrow \left[\frac{X(s)}{s} + x(0^+) \right] \end{aligned}$$

$$-\frac{x(t)}{t} \longleftrightarrow \int_s^{\infty} X(s) ds$$

$$x_1(t) * x_2(t) \longleftrightarrow X_1(s) X_2(s)$$

Problems

Solr

10. Convolution in s-domain

$$x_1(t) * x_2(t) \longleftrightarrow \frac{1}{2\pi} (x_1(s) * x_2(s))$$

* Initial Value Theorem

$$\lim_{t \rightarrow 0} x(t) = x(0) = \lim_{s \rightarrow \infty} s x(s)$$

Final Value Theorem

$$\lim_{t \rightarrow \infty} x(t) = x(\infty) = \lim_{s \rightarrow 0} s x(s)$$

Problems (1) - The Laplace transform of the signal $f(t)$ is $F(s) = \frac{(2s+1)}{s^2+4s+7}$.
then initial and final values of signal $f(t)$ are

Soln

$$F(s) = \frac{(2s+1)}{s^2+4s+7}$$

$$x(0) = \lim_{s \rightarrow \infty} s x(s)$$

$$= \lim_{s \rightarrow \infty} s \left(\frac{2s+1}{s^2+4s+7} \right)$$

$$= \lim_{s \rightarrow \infty} \frac{2[1 + 1/s]}{1 + 4/s + 7/s^2}$$

$$= \underline{\underline{2}}$$

$$x(\infty) = \lim_{s \rightarrow 0} s \left(\frac{2s+1}{s^2+4s+7} \right)$$

$$= 0$$

(2) The initial and final values of $X(s) = \frac{10s+10}{s(s+5)}$ for

(a) $\text{Re}(s) > 0$

(b) $\text{Re}(s) < -5$

Sol:-

$$X(s) = \frac{9s+10}{s(s+5)}$$

a) $X(0) = \lim_{s \rightarrow \infty} s \cdot \frac{9s+10}{s(s+5)}$

$$= \lim_{s \rightarrow \infty} \frac{s^2(9 + 10/s)}{s^2(1 + 5/s)} = 9$$

~~✗~~ $X(\infty) = \lim_{s \rightarrow 0} \frac{9s+10}{s(s+5)}$

$$= \frac{10}{5} = 2$$

(b) $X(0) = 2$

$X(\infty) = 9$

1) Find the inverse L-T of the signal $X(s) = \frac{9}{(s-1)(s+2)}$

Sol:-

$$X(s) = \frac{9}{(s-1)(s+2)}$$

$$= \frac{A}{s-1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

$$A=1, C=-9$$

$$\text{Since } A+B=0 \Rightarrow B=-A=-1$$

$$X(s) = \frac{1}{s-1} - \frac{1}{s+2} - \frac{3}{(s+2)^2}$$

$$x(t) = e^t u(t) - e^{-2t} u(t) - 3te^{-2t} u(t),$$

(4) A system is defined in its frequency response given as

$$H(j\omega) = \frac{j\omega + 4}{[6 + 5j\omega - \omega^2]}$$

(a) Find a diff eqn relating i/p and o/p response.

(b) Find its impulse response

(c) Find the step response

(d) Find the o/p response when i/p to the system is

$$x(t) = e^{-4t} u(t) - te^{-4t} u(t),$$

Sol: (a) $H(j\omega) = \frac{j\omega + 4}{(6 + 5j\omega - \omega^2)}$

$$H(s) = \frac{s+4}{s^2+5s+6} = \frac{y(s)}{x(s)}$$

$$s^2 y(s) + 5s y(s) + 6y(s) = s(x(s) + 4x(s))$$

Inverse L-T

$$\frac{d^2}{dt^2} y(t) + 5 \frac{d}{dt} y(t) + 6y(t) = \frac{d}{dt} x(t) + 4x(t)$$

(b) $H(\omega) = \frac{s+4}{(s+2)(s+3)}$

$$h(t) = \mathcal{L}^{-1}[H(s)]$$

$$\frac{A}{s+2} + \frac{B}{s+3}$$

$$A=2, B=-1$$

$$h(s) = \frac{A}{s+2} + \frac{B}{s+3}$$

$$= \frac{2}{s+2} - \frac{1}{s+3}$$

$$\boxed{h(t) = 2e^{-2t}u(t) - e^{-3t}u(t)}$$

(c)

$$\frac{Y(s)}{X(s)} = \frac{s+4}{(s+2)(s+3)}$$

$$\text{If } x(t) = u(t);$$

$$x(s) = 1/s$$

$$Y(s) = \frac{(s+4)}{s(s+2)(s+3)}$$

$$= \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$= \frac{2}{2s} - \frac{1}{s+2} + \frac{1}{3(s+3)}$$

$$\boxed{Y(s) = \frac{2}{3}u(t) - e^{-2t}u(t) + \frac{1}{3}e^{-3t}u(t)}$$

(d)

$$x(t) = e^{-4t}u(t) - te^{-4t}u(t)$$

$$X(s) = \frac{1}{s+4} - \frac{1}{(s+4)^2} = \frac{s+3}{(s+4)^2}$$

$$\frac{Y(s)}{X(s)} = \frac{s+4}{(s+2)(s+3)} \Rightarrow Y(s) = \frac{(s+4)}{(s+2)(s+3)} \times \frac{s+3}{(s+4)^2}$$

$$Y(s) = \frac{s+3}{(s+2)(s+4)}$$

$$Y(s) = \frac{s}{(s+2)(s+4)}$$

$$= \frac{A}{s+2} + \frac{B}{s+4}$$

$$= \frac{1}{2} \left[\frac{1}{s+2} - \frac{1}{s+4} \right]$$

$$y(t) = \frac{1}{2} [e^{-2t} - e^{-4t}] u(t)$$

15) A system whose voltage is given as a differential equation defined as $\frac{d^2v}{dt^2} + 2\frac{dv}{dt} + v = 0$ and the initial conditions

$V(0) = 10V$, $\frac{d}{dt}V(0) = 0V/s$ then the voltage $v(t)$ in the system is?

Sol:- $\frac{d^2v}{dt^2} + 2\frac{dv}{dt} + v = 0$

L-T on b.s

$$\left[s^2 V(s) - sV(0) - \frac{d}{dt}V(0) \right] + 2 \left[sV(s) - V(0) \right] + V(s) = 0$$

$$(s^2 + 2s + 1)V(s) - 10s + \cancel{\frac{d}{dt}V(0)} + 2(-10V) = 0$$

$$(s^2 + 2s + 1)V(s) - 10s - 20V = 0$$

$$V(s) = \frac{10s + 20}{(s+1)^2}$$

$$(s^2 + 2s + 1)V(s) = 10s + 20$$

$$V(s) = \frac{10(s+2)}{(s+1)^2} = \frac{10s}{(s+1)^2} + \frac{20}{(s+1)^2}$$

$$V(s) = \frac{10}{s+1} + \frac{10}{(s+1)^2}$$

$$\boxed{V(s) = 10e^{-t}u(t) + 10te^{-t}u(t)}$$

$$A=10$$

$$B=10$$

16/2/13

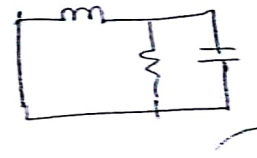
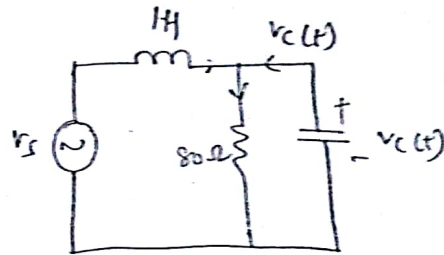
6. In a ckt $V_s=0$; $t>0$, initial conditions $v(0)=6V$, $\frac{dv(0)}{dt}=20V/s$.
Find the voltage $v_c(t)$ for $t>0$

Kcl:

$$i_L = \frac{1}{L} \int v(t) dt$$

$$i_C = C \frac{dv}{dt}$$

$$i_R = \frac{V}{R}$$



$$\frac{1}{1H} \int v_c(t) dt + \frac{v_c(t)}{80} - 25 \times 10^{-6} \frac{d}{dt} v_c(t) = 0$$

$$\frac{d^2}{dt^2} v_c(t) - \frac{1}{25 \times 10^{-6}} \times \frac{1}{80} \frac{d}{dt} v_c(t) - \frac{1}{25 \times 10^{-6}} v_c(t) = 0$$

$$\frac{d^2}{dt^2} v_c(t) - 500 \frac{d}{dt} v_c(t) - 40,000 v_c(t) = 0$$

L-T on b-s

$$\left[s^2 v(s) - s v(0) - \frac{d}{dt} v(0) \right] - 500 \left[s v(s) - v(0) \right] - 40,000 v(s) = 0$$

$$\left[s^2 - 500s - 40,000 \right] v(s) - s(6V) - \frac{d}{dt} 20 + 500 \times 6 = 0$$

$$(s^2 - 500s - 40,000) v(s) - 6s + 2700 = 0$$

$$v(s) = \frac{6s - 2700}{(s^2 - 500s - 40,000)}$$

$$= \frac{6s - 2700}{(s - 570.1)(s + 70.15)}$$

$$= \frac{A}{(s - 570.1)} + \frac{B}{(s + 70.15)}$$

$$\Rightarrow A = 1.125, \quad B = 0.65$$

$$\leq \frac{1.125}{(s - 570.1)} + \frac{0.65}{(s + 70.15)}$$

$$v_c(t) = 1.125e^{570.1t} + 0.65e^{-70.15t}$$

7. Transfer function $H(s)$ in the given circuit and also find its impulse response.

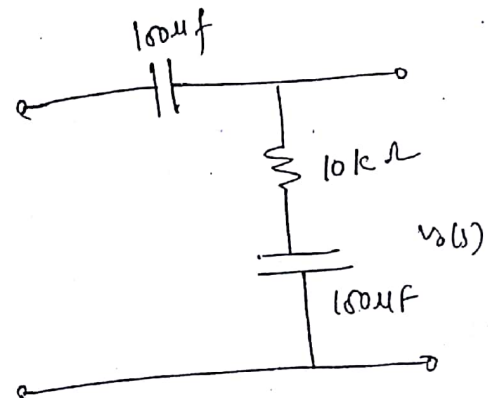
$$100\mu f \Rightarrow \frac{1}{s \times 100 \times 10^{-6}} = \frac{10^4}{s}$$

$$10k\Omega \rightarrow 10k$$

$$V_o(s) = \frac{V_i(s) \times \left(10k + \frac{10^4}{s}\right)}{\left(\frac{10^4}{s} + \frac{10^4}{s} + 10k\right)}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{10ks + 10^4}{10^4 + 10^4 + 10ks} = \frac{s+1}{s+2}$$

$$H(s) = \frac{s+1}{s+2} = \frac{s+2}{s+2} - \frac{1}{s+2}$$



$$H(s) = 1 - \frac{1}{s+2}$$

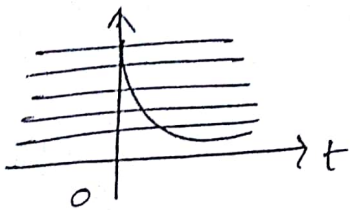
$$h(t) = \delta(t) - e^{-2t}u(t)$$

Bilateral d.T (BLT)

$$\text{BLT } \{x(t)\} = X(s)$$

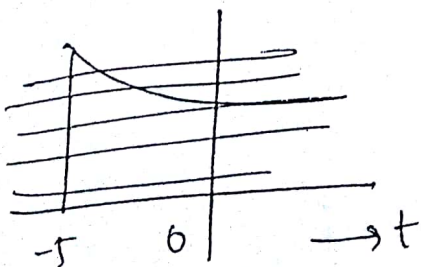
$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

1. $x(t) = e^{-2t} u(t)$



$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} e^{-2t} u(t) e^{-st} dt \\ &= \int_0^{\infty} e^{-t(s+2)} dt = \frac{1}{s+2} \end{aligned}$$

2. $x(t) = e^{-2(t+5)} u(t+5)$



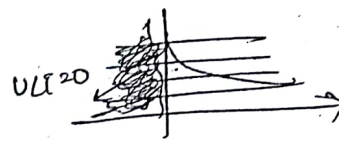
$$X(s) = e^{5s} \times \frac{1}{s+2}$$

Unilateral L-T (ULT)

$$\text{ULT } \{x(t)\} = X(s)$$

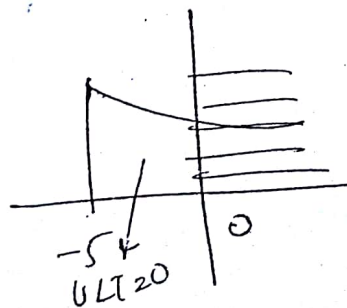
$$X(s) = \int_0^{\infty} x(t) e^{-st} dt$$

$x(t) = e^{-2t} u(t)$



$$\begin{aligned} X(s) &= \int_0^{\infty} e^{-2t} u(t) e^{-st} dt \\ &= \int_0^{\infty} e^{-(s+2)t} dt = \frac{1}{s+2} \end{aligned}$$

2. $x(t) = e^{-2(t+5)} u(t+5)$



For ULT; $x(t) = e^{-2(t+5)} u(t)$

$$= e^{-10} \times e^{-2t} u(t)$$

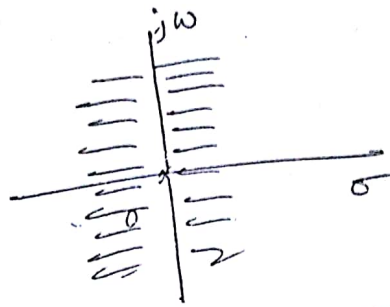
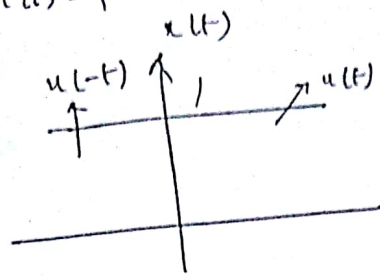
$$X(s) = \frac{e^{-10}}{s+2}$$

* find

(1)

* Find A.L.T and V.L.T for the given signals.

(1) $x(t) = 1$



Since common ROC is present A.L.T doesn't exist

for V.L.T

$$x(t) = u(t)$$

$$X(s) = \frac{1}{s}$$

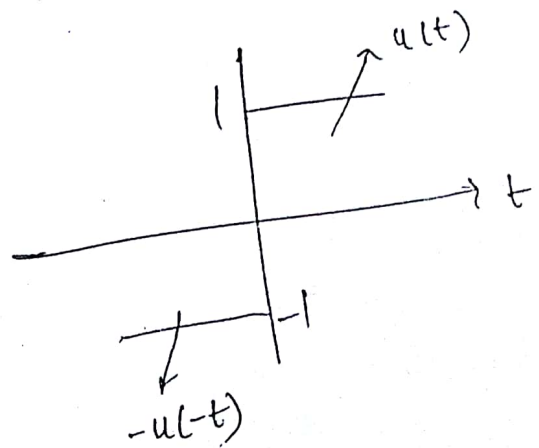
$$\boxed{\text{Re}(s) > 0}$$

(2) $x(t) = \text{sgn}(t)$
 $= u(t) - u(-t)$

$$\text{sgn}(t) = 1; t > 0$$

$$= -1; t < 0$$

A.L.T doesn't exist



for V.L.T

$$x(t) = u(t)$$

$$X(s) = 1/s$$

$$x(t) = e^{5t} u(t)$$

$$= e^{5t} u(t) + e^{-5t} u(-t)$$

∴ BLT doesn't exist

For ULT

$$x(t) = e^{5t} u(t)$$

$$X(s) = \frac{1}{s-5}$$

$$\boxed{\text{Re}(s) > 5}$$

* Find the Inverse Laplace transform for the signal

$$X(s) = \frac{1}{s^2 + 5s + 6} \text{ for condition}$$

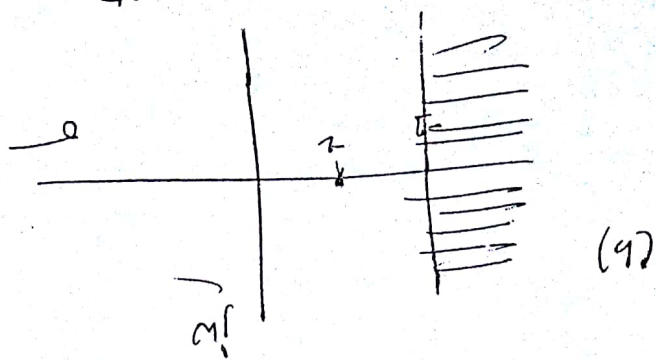
- (a) ULT (b) $\text{Re}(s) < -3$ (c) $\text{Re}(s) < -2$

$$X(s) = \frac{1}{(s+2)(s+3)}$$

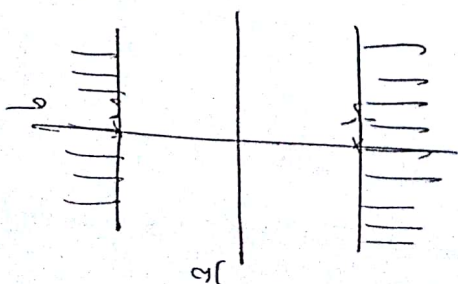
$$= \frac{A}{s+2} + \frac{B}{s+3}$$

$$= \frac{1}{s+2} - \frac{1}{s+3}$$

(a) ULT $x(t) = e^{-2t} u(t) - e^{-3t} u(t)$



$$x(t) = -e^{-2t} u(t) + e^{-3t} u(-t)$$

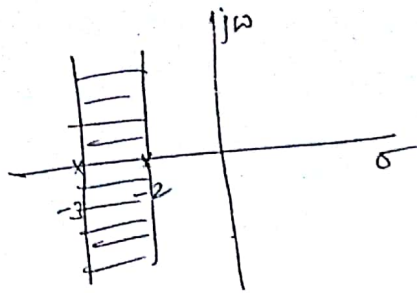


(c)

Sol:-

* In

(c)



$$x(t) = -e^{-2t}u(t) - e^{-3t}u(t)$$

* The Inverse L-T of VLT $x(s) = \ln \left\{ \frac{(s+a)}{(s+b)} \right\}$

Sol:- $x(s) = \ln \left\{ \frac{(s+a)}{(s+b)} \right\}$

$$\left(\frac{d}{ds} x(s) \right) \longleftrightarrow -tx(t) \longleftrightarrow \frac{d}{ds} x(s)$$

$$x(s) = \ln(s+a) - \ln(s+b)$$

$$= \frac{d}{ds} [\ln(s+a) - \ln(s+b)]$$

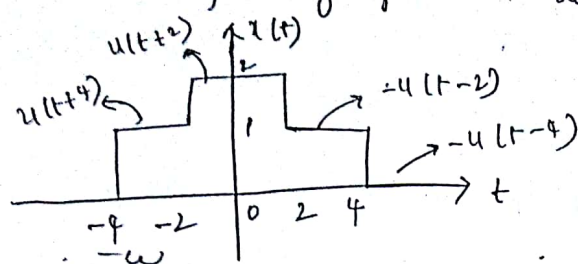
$$= \frac{1}{s+a} - \frac{1}{s+b}$$

$$-tx(t) = e^{-at}u(t) - e^{-bt}u(t)$$

$$x(t) = \frac{1}{t} [e^{-bt}u(t) - e^{-at}u(t)]$$

L.T for waveforms

(1) find the L.T for the following signals and also find ROC.

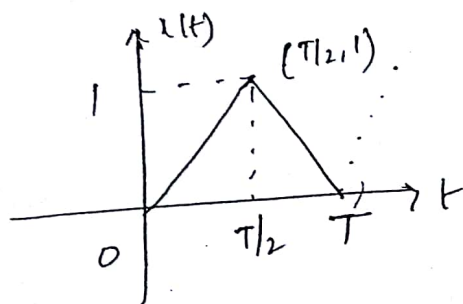


$$x(t) = u(t+4) + u(t+2) - u(t-2) - u(t-4)$$

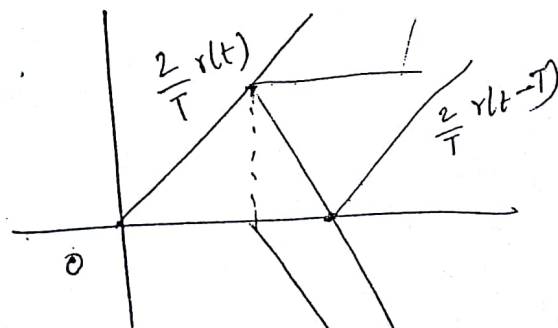
$$X(s) = \frac{e^{4s}}{s} + \frac{e^{2s}}{s} - \frac{e^{-2s}}{s} - \frac{e^{-4s}}{s}$$

ROC is entire s-plane except for $s=0, s=\infty$

(2)



$$x(t) = \frac{2}{T} r(t) + \frac{2}{T} r(t-T/2) + \frac{2}{T} r(t-T)$$



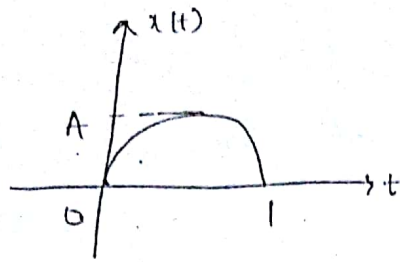
$$\frac{2}{T} r(t-T/2) + \frac{2}{T} r(t-T)$$

$$X(s) = \frac{2}{T} \cdot \frac{1}{s} - \frac{4}{T} \cdot e^{-sT/2} \times \frac{1}{s} + \frac{2}{T} e^{-sT} \times \frac{1}{s}$$

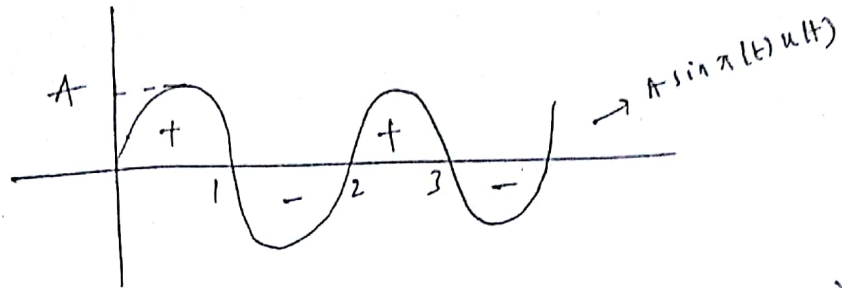
$$X(s) = \frac{2}{Ts} [1 - e^{-sT/2}]^2$$

(4)

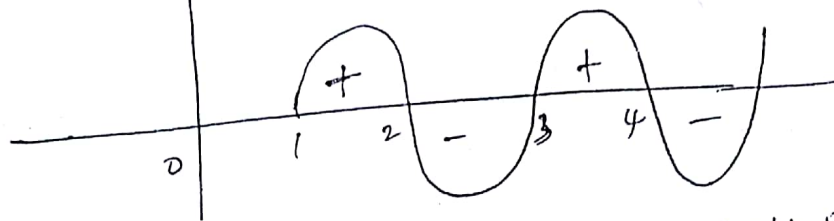
(3)



$$T=2, \omega = \frac{2\pi}{T} = \pi \text{ rad/s}$$



$$A \sin \pi(t-1)u(t-1)$$

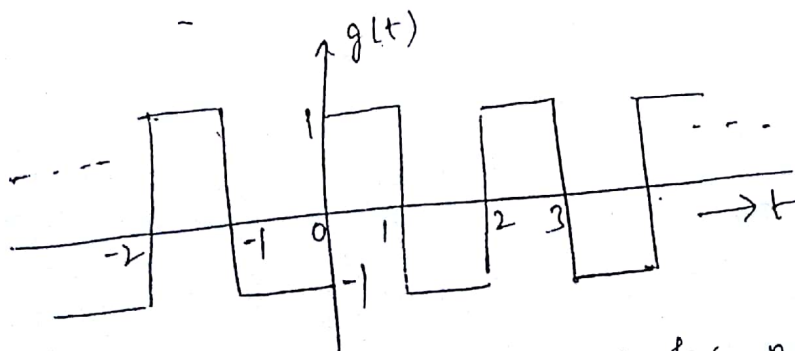


$$x(t) = A \sin \pi t u(t) + A \sin \pi(t-1) u(t-1)$$

$$X(s) = A \cdot \frac{\pi}{s^2 + \pi^2} + e^{-s} \left[\frac{A\pi}{s^2 + \pi^2} \right]$$

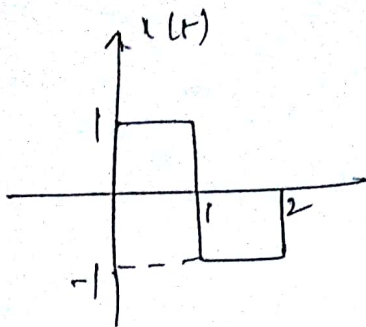
$$X(s) = \frac{A\pi}{s^2 + \pi^2} [1 + e^{-s}]$$

(4)



NOTE 1 RLT cannot be obtained for periodic signals since there is no common ROC b/w the poles. So, ~~RLT~~ VLT is obtained for the periodic signals.

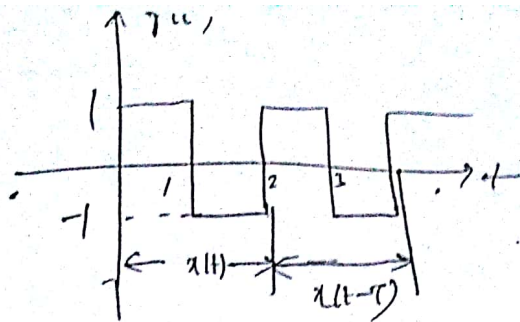
V.L.T



$$x(t) = u(t) - 2u(t-1) + u(t-2)$$

$$= \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s}$$

$$G(s) = \frac{1}{(1-e^{-2s})} \cdot \left[\frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s} \right]$$



$$y(t) = x(t) + x(t-T) + x(t-2T) + \dots$$

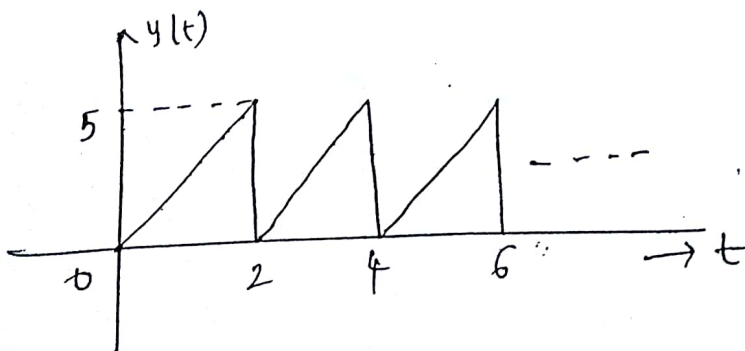
∞ -T on b.s

$$G(s) = X(s) + e^{-sT}X(s) + e^{-2sT}X(s) + \dots$$

$$= X(s) [1 + e^{-sT} + e^{-2sT} + \dots]$$

$$G(s) = \frac{X(s)}{(1-e^{-sT})}$$

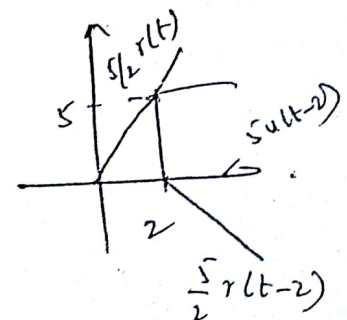
(5)



$$y(t) = \frac{5}{2}r(t) - \frac{5}{2}r(t-2) - 5u(t-2)$$

$$Y(s) = \frac{5}{2} \cdot \frac{1}{s^2} - \frac{5}{2} \cdot \frac{e^{-2s}}{s^2} - \frac{5e^{-2s}}{s}$$

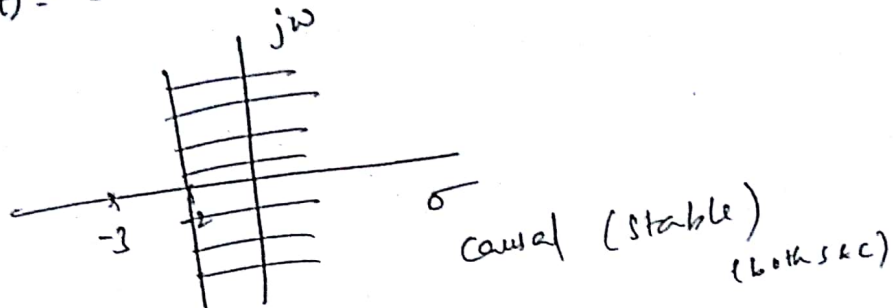
$$Y(s) = G(s) = \frac{1}{1-e^{-2s}} \left[\frac{5}{2s^2} [1-e^{-2s}] - \frac{5e^{-2s}}{s} \right]$$



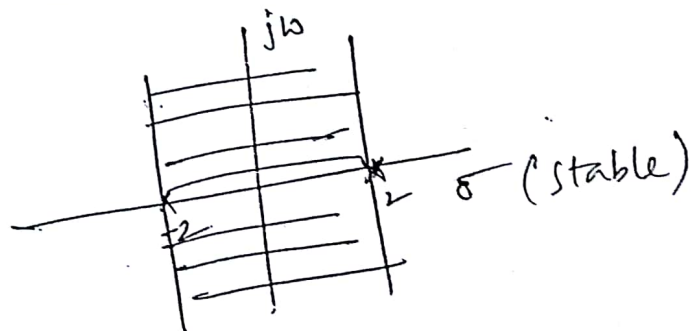
System characterisation using A.T:

* Causality: A system response $H(s)$ is said to be only causal only if its ROC is the region to the right of the right most pole.

$$\text{Eg 1.} h(t) = e^{-2t}u(t) + e^{-3t}u(t)$$



$$\begin{aligned} \text{(2)} \quad h(t) &= e^{-2t} \rightarrow \text{Non-causal} \\ &= e^{-2t}u(t) + e^{2t}u(-t) \end{aligned}$$



* Stability: For the system response to be only stable the ROC must include its imaginary axis.

* Both stable & causal: For the system response to be both stable and causal all the poles must lie on the left half of s-plane

1) A system response is characterised as $H(s) = \frac{2-7.5s}{s^2+5s+4}$. Find the conditions for which the system becomes only stable, only causal and both s & c. by neatly sketching its ROC.

Sol:-

$$H(s) = \frac{2-7.5s}{s^2+5s+4}$$

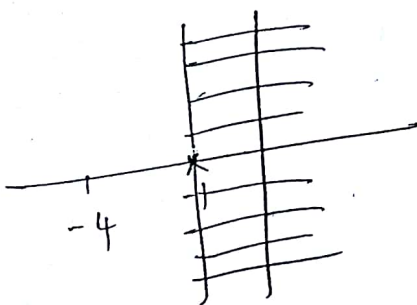
$$s^2+5s+4$$

$$s(s+4) + 1(s+4)$$

$$(s+1)(s+4)$$

$$s = -1, -4$$

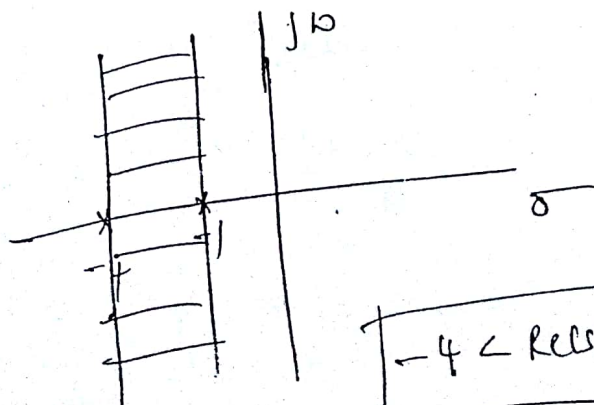
① $s = -1, -4$



$$\text{Re}(s) > -1$$

System is stable and causal

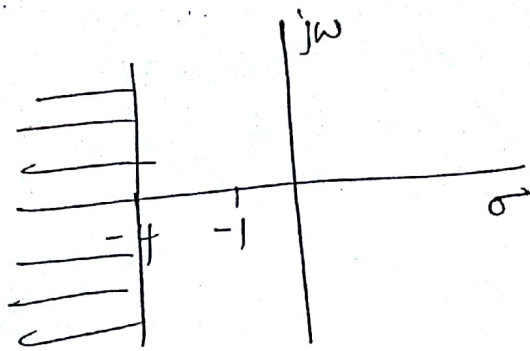
②



$$-4 < \text{Re}(s) < -1$$

system is neither stable nor causal

③



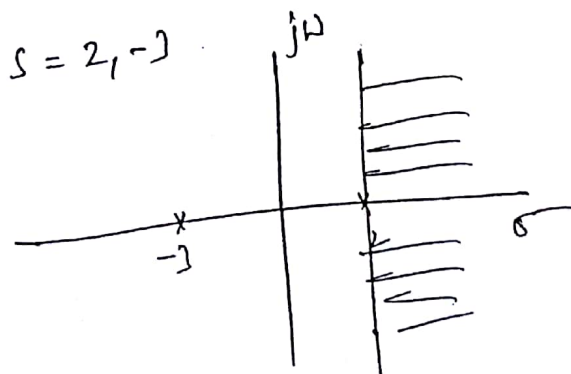
$$\boxed{\operatorname{Re}(s) < -4}$$

Neither causal nor stable

(2) $H(s) = \frac{1-2s}{s^2+s-6}$

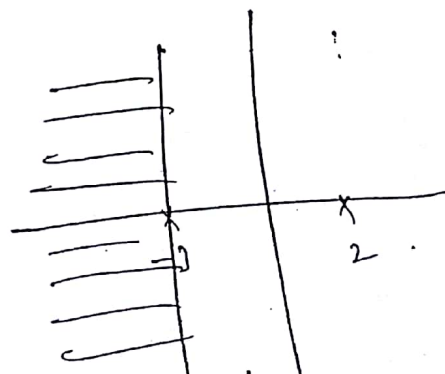
$$\begin{aligned} & s^2 + s - 6 \\ & s^2 + 3s - 2s - 6 \\ & s(s+3) - 2(s+3) \\ & (s-2)(s+3) \\ & s = +2, -3 \end{aligned}$$

①



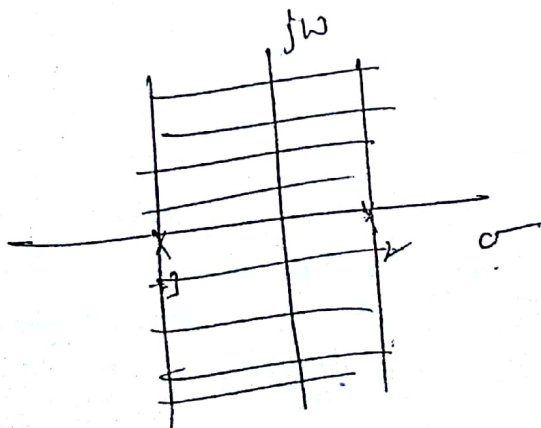
$\operatorname{Re}(s) > 2$ (causal)

②



$\operatorname{Re}(s) < -3$
neither stable
nor causal

②

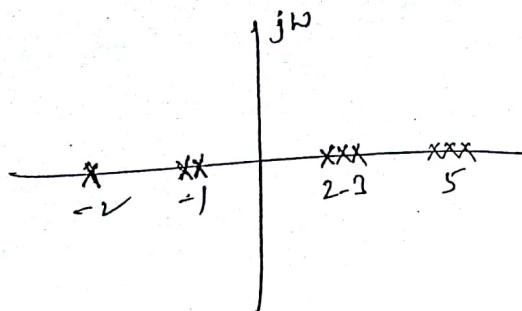


(stable but not
causal)

(3) find the impulse response for the stable transfer func. of

$$H(s) = \frac{s+1}{(s-5)^2(s-2)^2(s+1)^2(s+2)}$$

Soln



$$H(s) = \frac{(s+1)}{(s+1)^2(s+2)}$$

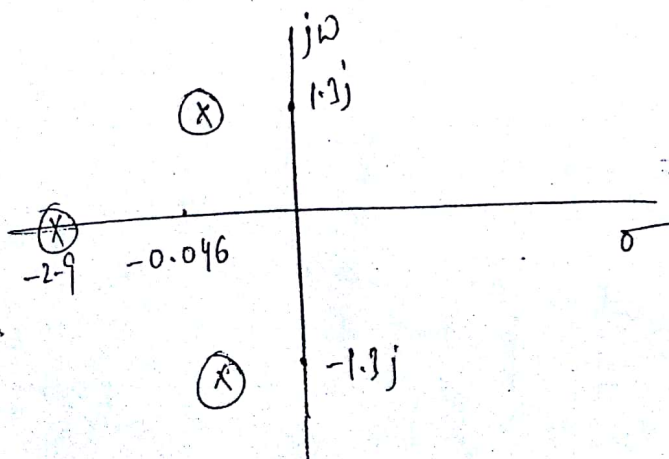
$$= \frac{-1}{s+1} + \frac{2}{(s+1)^2} + \frac{1}{s+2}$$

$$h(t) = -e^{-t}u(t) + 2te^{-t}u(t) + e^{-2t}u(t)$$

(4) find whether the system response $H(s) = \frac{1}{(s^2+3s+5)}$

$$H(s) = \frac{1}{s^2+3s+5}$$

$$s = -2.9, -0.04 \pm 1.3j$$

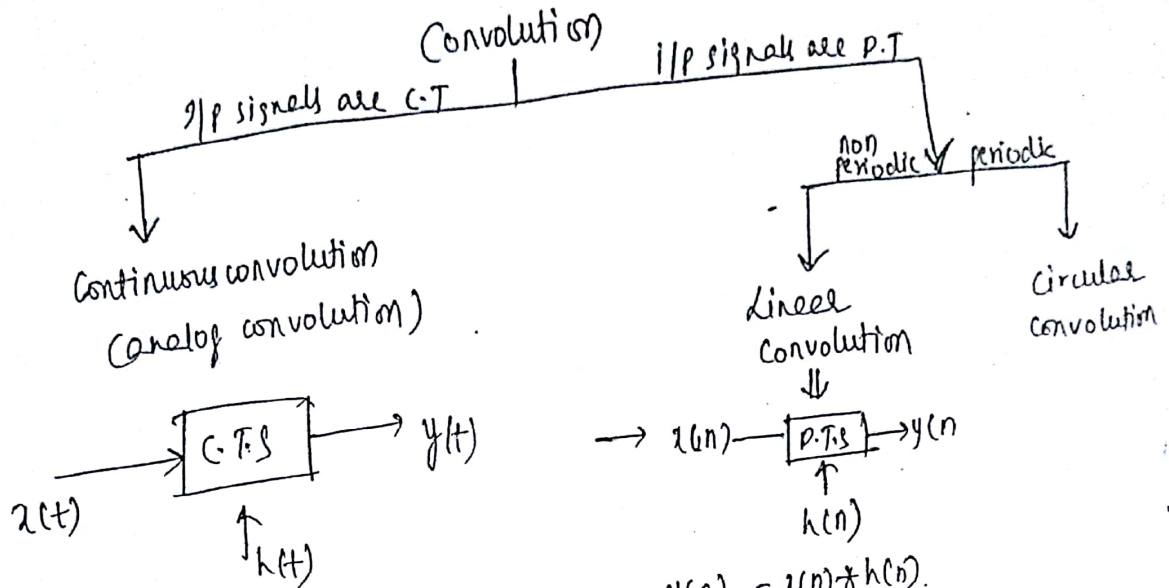


all poles
lie left
side of
s-plane
so, system
is stable

18/2/13

Convolution: The process of obtaining the system response by continuous mixing of the i/p response with its impulse response.

Classification:

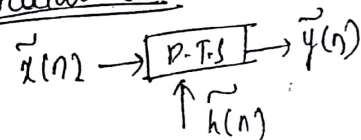


$$\begin{aligned}
 y(t) &= x(t) * h(t) \\
 &= \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \quad (\text{or}) \\
 &= \int_{-\infty}^{\infty} x(t-\tau) h(\tau) d\tau.
 \end{aligned}$$

$x(t) \rightarrow$ i/p response
 $y(t) \rightarrow$ o/p response
 $h(t) \rightarrow$ impulse response

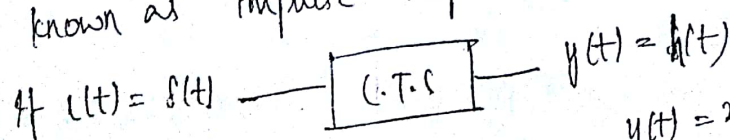
$$\begin{aligned}
 y(n) &= x(n) * h(n) \\
 &= \sum_{k=-\infty}^{\infty} x(k) h(n-k) \quad \text{or} \\
 &= \sum_{k=-\infty}^{\infty} x(n-k) h(k)
 \end{aligned}$$

Circular conv.



$$\begin{aligned}
 \tilde{y}(n) &= \tilde{x}(n) \otimes \tilde{h}(n) \\
 &= \sum_{k=-\infty}^{\infty} x(k) h(n-k), \text{mod } N \\
 &= \sum_{k=-\infty}^{\infty} x(n-k) h(k), \text{mod } N
 \end{aligned}$$

Impulse: If the i/p to the system is an impulse function the response is known as impulse response.



$$\begin{aligned}
 y(t) &= x(t) * h(t) \\
 &= \delta(t) * h(t) = h(t)
 \end{aligned}$$

Step response $s(t)$

If $x(t) = u(t)$ then
 $y(t) = s(t)$

$$y(t) = x(t) * h(t) \\ = \int_{-\infty}^{\infty} x(t-\tau) h(\tau) d\tau$$

$$s(t) = \int_{-\infty}^{\infty} u(t-\tau) h(\tau) d\tau$$

$$** \boxed{s(t) = \int_{-\infty}^t h(\tau) d\tau} \quad \text{--- (1)}$$

$$** \boxed{h(t) = \frac{d}{dt} s(t)} \quad \text{--- (2)}$$

NOTE If i/p signal $h(t)$ is a causal signal then step response

$$\boxed{s(t) = \int_0^t h(\tau) d\tau}$$

Problem

c1) The unit step response of a system is defined as $s(t) = e^{-2t} u(t)$
then the impulse response to the system is?

Soln

$$s(t) = e^{-2t} u(t)$$

$$h(t) = \frac{d}{dt} e^{-2t} u(t)$$

$$= e^{-2t} \times -2 u(t) + e^{-2t} \delta(t)$$

$$= -2e^{-2t} u(t) + 1 \times \delta(t)$$

$$\boxed{h(t) = \delta(t) - 2e^{-2t} u(t)}$$

(2) The ii
given
the
Soln

12) The impulse response of a system under '0 initial conditions is given as $h(t) = e^{-t}u(t) + e^{-2t}u(t)$ then the (input) response of the system is?

Soln-

$$h(t) = e^{-t}u(t) + e^{-2t}u(t)$$

$$s(t) = \int_0^t e^{-\tau}u(\tau) + \int_0^t e^{-2\tau}u(\tau) d\tau$$

$$= \left[\frac{e^{-\tau}}{-1} \right]_0^t + \left[\frac{e^{-2\tau}}{-2} \right]_0^t$$

$$= 1.5 - e^{-t}u(t) - \frac{1}{2}e^{-2t}u(t)$$

(or)

$$y(t) = x(t) * h(t)$$

L.T on b.s.

$$Y(s) = X(s) \times H(s)$$

$$Y(s) = \frac{1}{s} \left[\frac{1}{s+1} + \frac{1}{s+2} \right]$$

$$= \left[\frac{1}{s(s+1)} + \frac{1}{s(s+2)} \right]$$

$$\neq \frac{-1}{s(s+1)} + \frac{1}{s} + \frac{2}{s(s+2)}$$

$$Y(s) = \frac{3}{2s} - \frac{1}{s+1} - \frac{1}{2(s+2)}$$

$$\frac{(s+2) + (s+1)}{s}$$

$$\frac{1}{s} \left[\frac{A}{s+1} + \frac{B}{s+2} \right] = 1$$

$$\frac{1}{s} (A(s+2) + B(s+1)) = 1$$

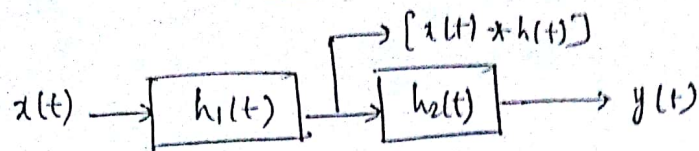
$$A(1) = 1$$

$$\frac{1}{-2} (A(-1) + B(0)) = 1$$

$$B(-1) = -2$$

$$(B = 2)$$

(1) Systems connected in cascade

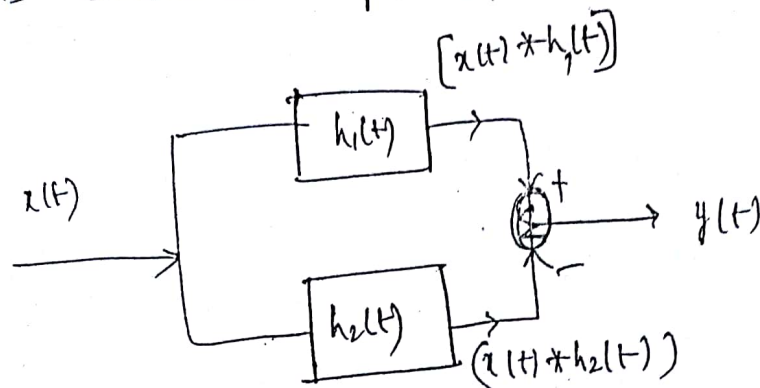


$$y(t) = [x(t) * h_1(t)] * h_2(t)$$

$$= x(t) * [h_1(t) * h_2(t)]$$

- satisfies associative law of convolution
- satisfies commutative law of convolution

(2) systems connected in parallel

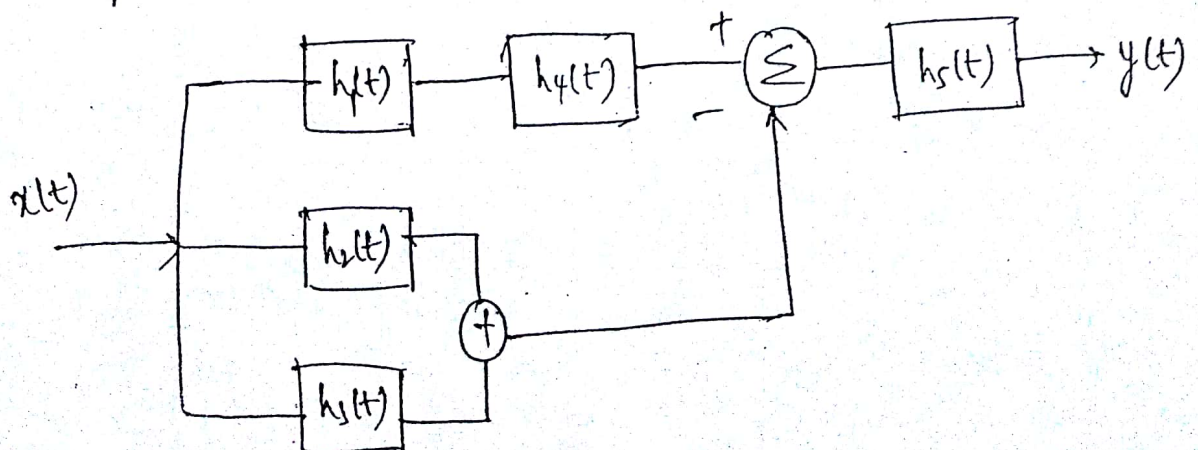


$$y(t) = [x(t) * h_1(t)] + [x(t) * h_2(t)]$$

$$= x(t) * [h_1(t) + h_2(t)]$$

- satisfies distributive law.

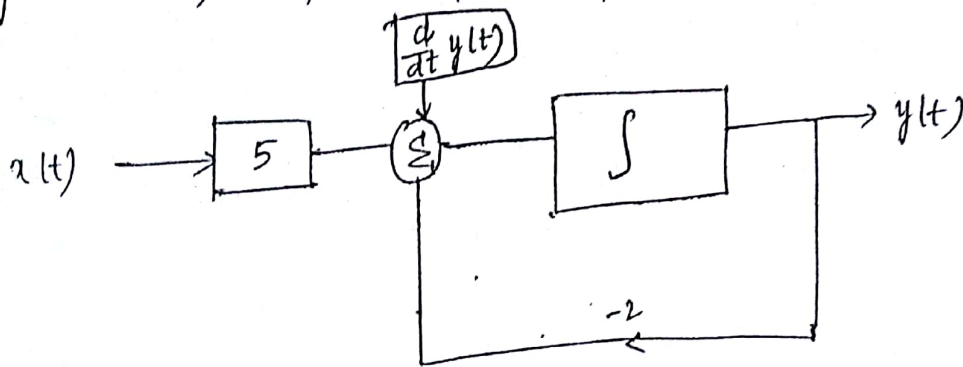
* Find the equation for overall system response



NOTE: The continuous time systems formulates differential equation and discrete time systems formulates difference equations.

Formulation of Differential Equations:

(i) formulate the differential eqn for the given system and find whether they are LTI, LTV, NLTI, NLTV, Reversible or Irreversible.



$$\frac{d}{dt} y(t) = 5x(t) - 2y(t)$$

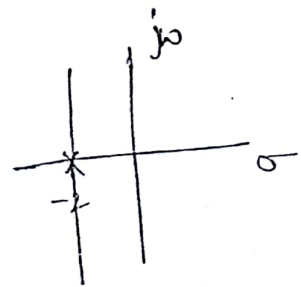
$$\frac{d}{dt} y(t) + 2y(t) = 5x(t)$$

(1) LTI (2) Irreversible

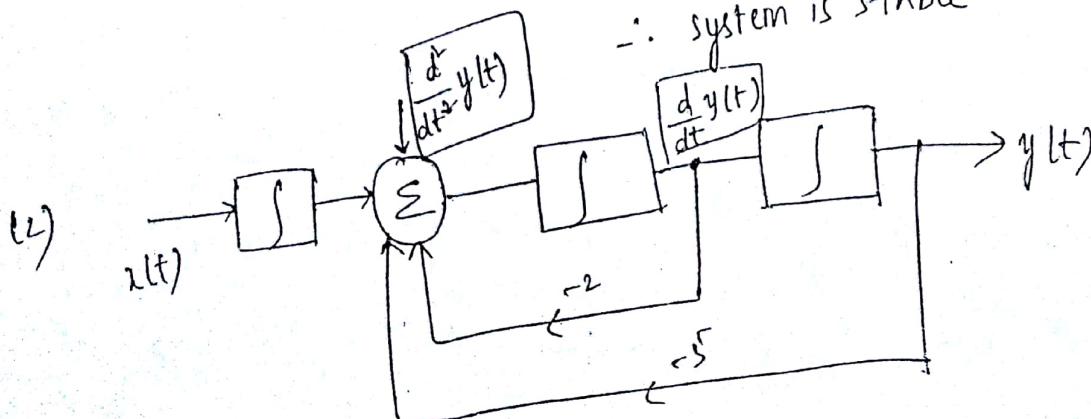
→ L.T on both sides

$$sY(s) + 2Y(s) = 5X(s)$$

$$\frac{Y(s)}{X(s)} = \frac{5}{s+2}$$



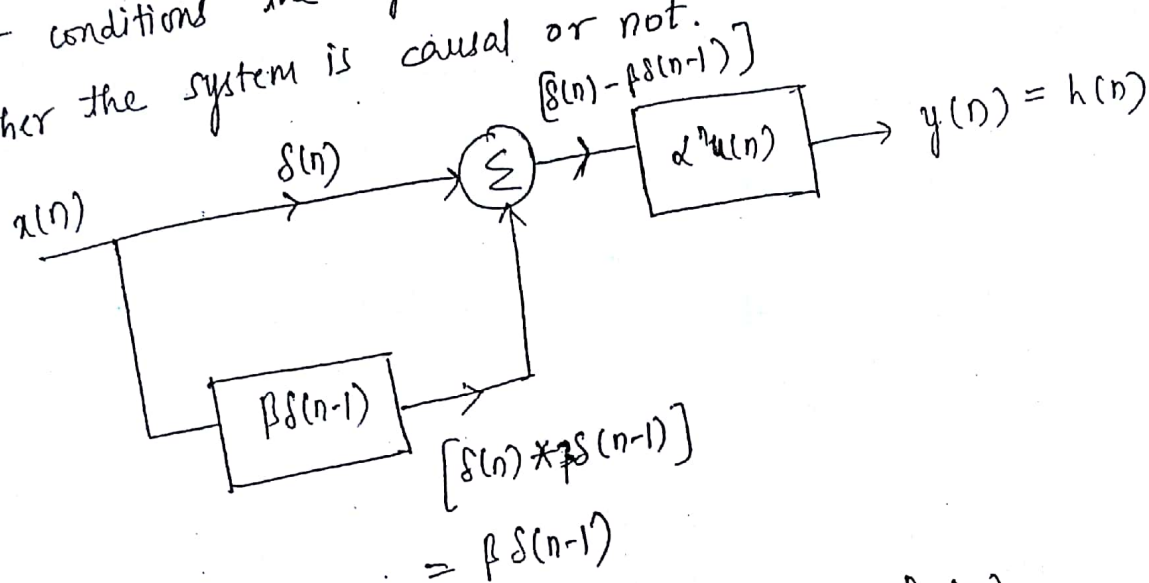
∴ system is stable.



$$\begin{aligned} &\rightarrow [x(t) * h_1(t)] * h_4(t) \\ &= x(t) * [h_1(t) * h_4(t)] \\ &\rightarrow x(t) * (h_2(t) + h_3(t)) \end{aligned}$$

$$y(t) = x(t) * h_5(t) * \{ h_1(t) * h_4(t) - [h_2(t) + h_3(t)] \}$$

* Find the impulse response of the given system. find under what conditions the system becomes stable? and also verify whether the system is causal or not.



$$\begin{aligned} h(n) &= [\delta(n) - \beta \delta(n-1)] * \alpha^n u(n) \\ &= [\delta(n) * \alpha^n u(n)] - [\beta \delta(n-1) * \alpha^n u(n)] \end{aligned}$$

$$h(n) = \alpha^n u(n) - \beta \alpha^{(n-1)} u(n-1)$$

(i) $|\alpha| < 1$ [system becomes stable]
 $\rightarrow$ For the system response where $|\alpha| < 1$ becomes exponentially decaying so the system becomes stable.
 $\therefore$ The system is absolutely causal $n \rightarrow 0$ to ∞

$$y(t) = 5x(t) - 3p(t) \rightarrow y(t) \\ = 5x(t) - 3 \int y(t) dt - 5 \int p(t) dt$$

D.w.r.t on b.s

$$\frac{d}{dt} y(t) = 5 \frac{d}{dt} x(t) - 3 \frac{d}{dt} \int y(t) dt - 5 \int p(t) \cdot dt$$

D.w.r.t on b.s

$$\frac{d^2}{dt^2} y(t) + 3 \frac{d}{dt} y(t) + 5y(t) = 5 \frac{d^2}{dt^2} x(t)$$

$$y(t) \left[\frac{d^2}{dt^2} + 3 \frac{d}{dt} + 5 \right] = 5 \frac{d^2}{dt^2} x(t)$$

d.T on both sides

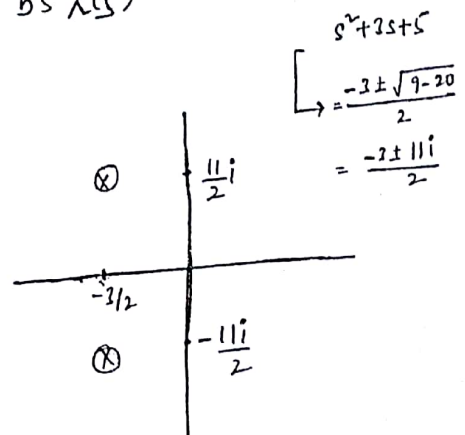
(1) LTI

(2) irreversible

$$Y(s) [s^2 + 3s + 5] = 5s^2 X(s)$$

$$\frac{Y(s)}{X(s)} = \frac{5s^2}{s^2 + 3s + 5}$$

→ (stable)



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Graphical method of Convolution:

(1) Find the limits of output response using limits rule stated as:
sum of lower limits < t < sum of upper limits.

(2) Graphically obtain $x(\tau) h(t-\tau)$ 'or' $x(t-\tau) h(\tau)$.

(3) keeping τ as constant and varying different values of t
obtain $\int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$ or $\int_{-\infty}^{\infty} x(t-\tau) h(\tau) d\tau$.

$$\frac{d^2}{dt^2} y(t) = \int x(t) dt - 2 \frac{d}{dt} y(t) - 5y(t)$$

Diff. on b.s

$$\frac{d^3}{dt^3} y(t) = 2 \frac{d^2}{dt^2} y(t) + 5 \frac{d}{dt} y(t) = x(t)$$

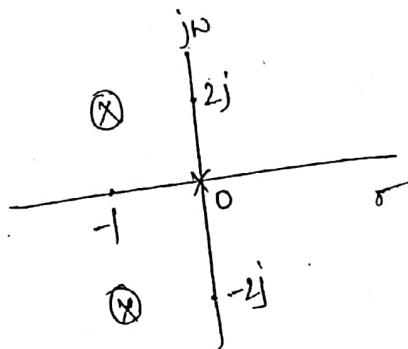
(1) d.I (2) irreversible

$$s^3 y(s) + 2s^2 y(s) + 5s y(s) = X(s)$$

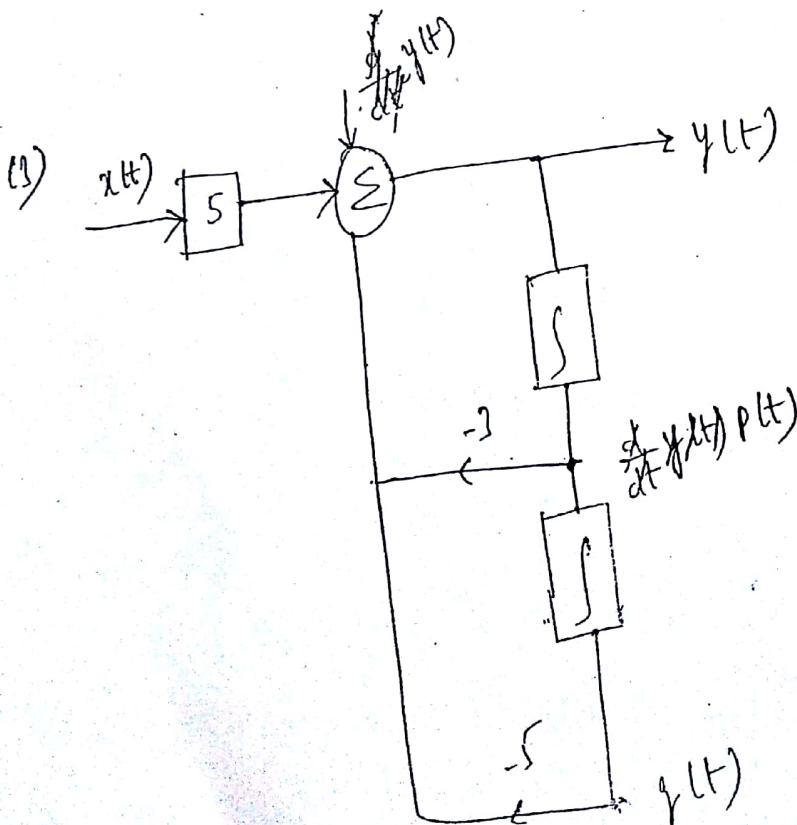
$$Y(s) [s^3 + 2s^2 + 5s] = X(s)$$

$$\frac{Y(s)}{X(s)} = H(s) = \frac{1}{s^3 + 2s^2 + 5s} = \frac{1}{s(s^2 + 2s + 5)}$$

$$s=0, s=-1 \pm 2j$$

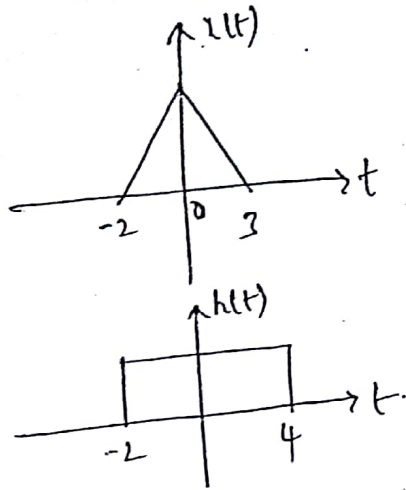


∴ This system is marginally stable



(4) Summarise all the results of t and sketch the o/p response $y(t)$.

* For the given signals $x(t)$ and $h(t)$, find the limits of o/p response and length of the o/p convolution



$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

$$\text{Limit of o/p response} = \boxed{-4 < t < 7}$$

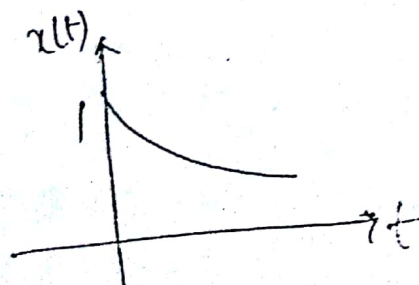
$$\text{o/p response} = 7 - (-4) = 11$$

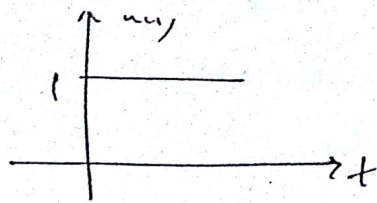
(2) Find the o/p response $x(t) = e^{-at} u(t)$; $a > 0$, $h(t) = u(t)$ using graphical method.

Sol:

$$x(t) = e^{-at} u(t) ; a > 0$$

$$h(t) = u(t)$$





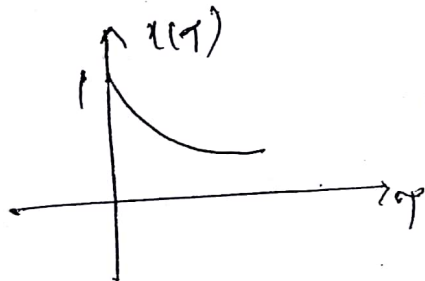
limits of o/p response

$$0 < t < \infty$$

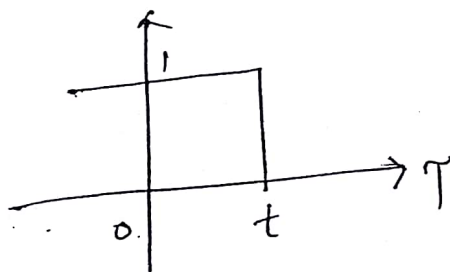
$$0 < t < \infty$$

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

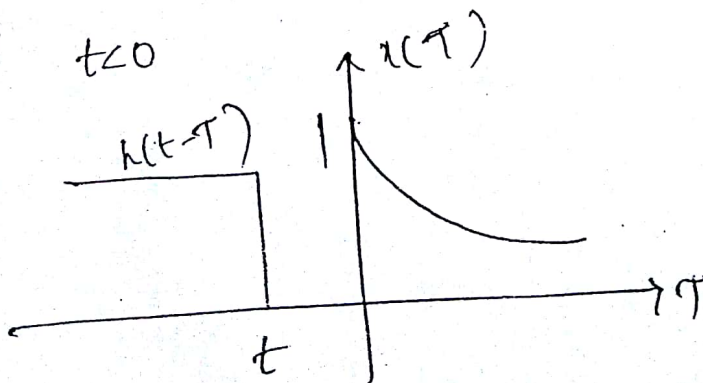


$$h(t-\tau) = u(t-\tau)$$



Case (i)

$$t < 0$$



Since no common response

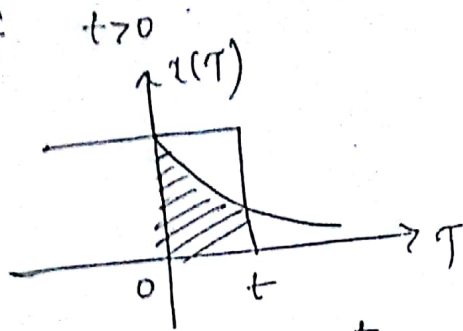
$$\text{So } y(t) = 0$$

Case

* f

So/1-

Case (ii):



$$y(t) = \int_0^t e^{-a\tau} x(\tau) u(t-\tau) d\tau$$

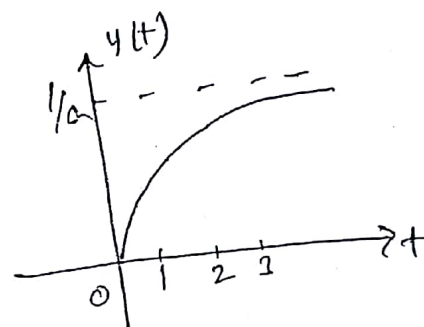
$$= \left[\frac{e^{-a\tau}}{-a} \right]_0^t$$

$$= \frac{1}{a} (1 - e^{-at});$$

Summarise

$$y(t) = 0; t < 0$$

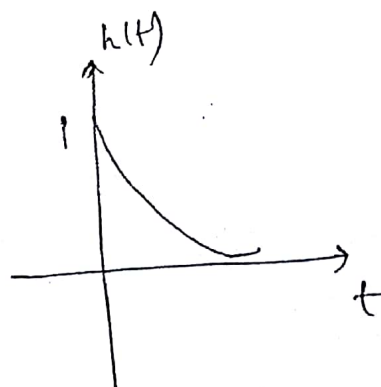
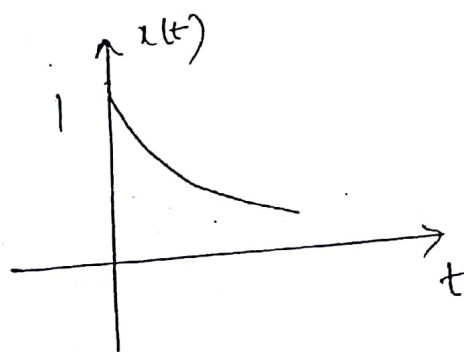
$$= \frac{1}{a} (1 - e^{-at}); t > 0$$



* Find the o/p response when $e^{-at} u(t)$ convolved with itself

Soln-

$$x(t) = e^{-at} u(t); a > 0 \quad h(t) = x(t)$$



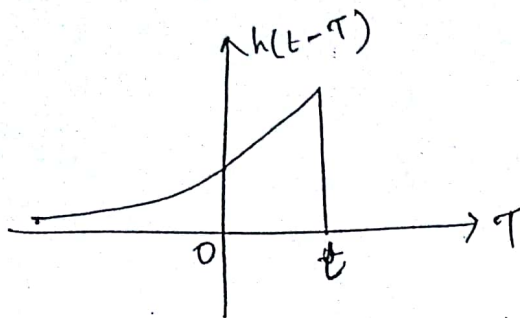
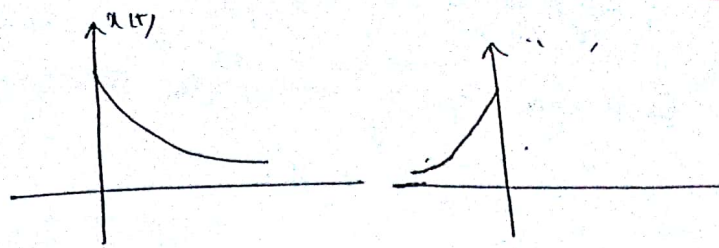
$$0 < t < \infty + \infty$$

$$\boxed{0 < t < \infty}$$

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

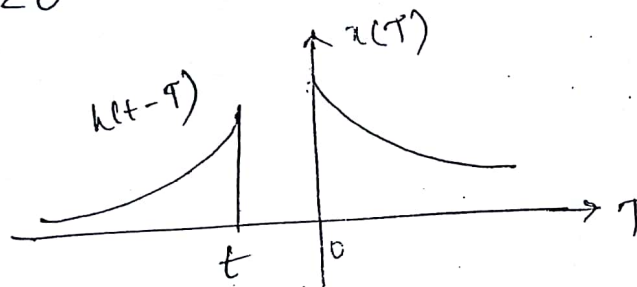
$$h(t-\tau) = h[-(\tau-t)]$$



Find

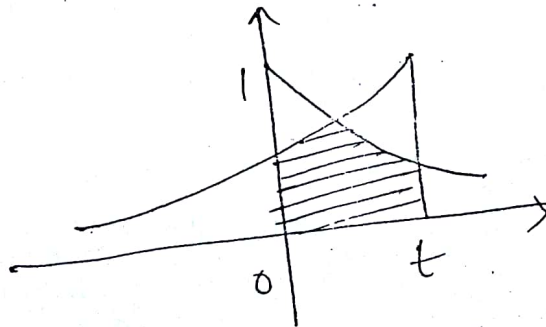
Soln

Case (i) $t < 0$



Since no common response
 $y(t) = 0$

Case (ii) $t > 0$



$$y(t) = \int_0^t e^{-at} u(\tau) e^{-a(t-\tau)} u(t-\tau) d\tau$$

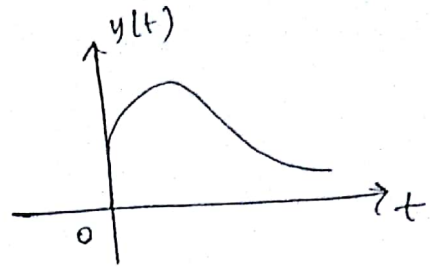
$$= \int_0^t e^{-at} \cdot e^{-at} \cdot e^{a\tau} d\tau$$

$$y(t) = \int_0^t e^{-a\tau} d\tau$$

$$= t \cdot e^{-at}; t > 0$$

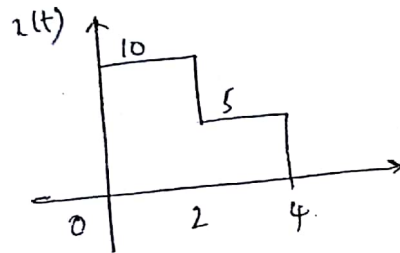
$$y(t) = 0; t < 0$$

$$= t e^{-at}; t > 0$$

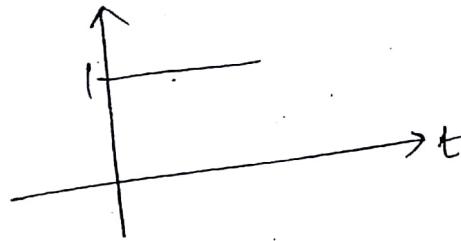


Find the o/p response when the signal convolved with $h(t) = u(t)$

Sol:-

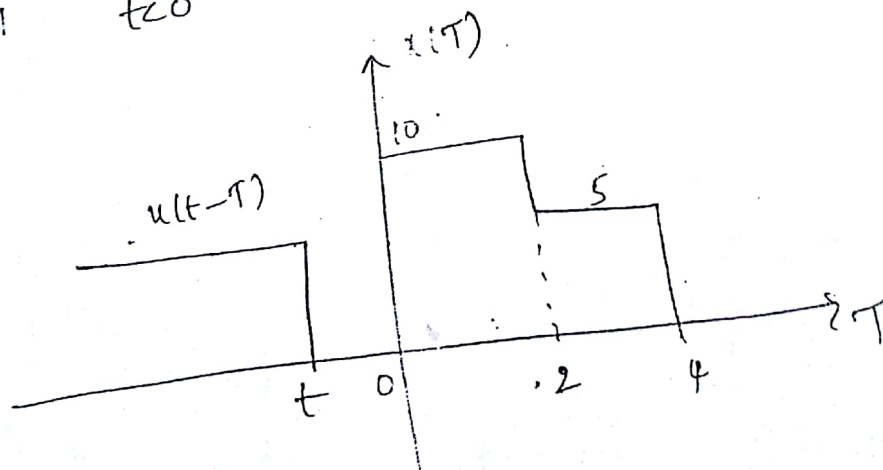


$$h(t) = u(t)$$



Case (i):

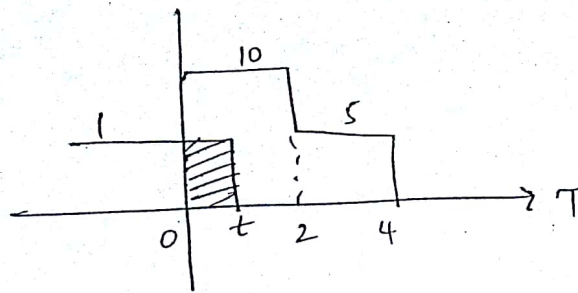
$t < 0$



$$y(t) = 0$$

Case (i):

$$0 < t < 2$$

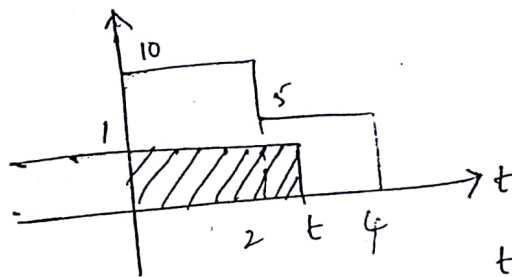


$$y(t) = \int_0^t 10 \times 1 d\tau$$

$$= 10t$$

Case (ii):

$$2 < t < 4$$

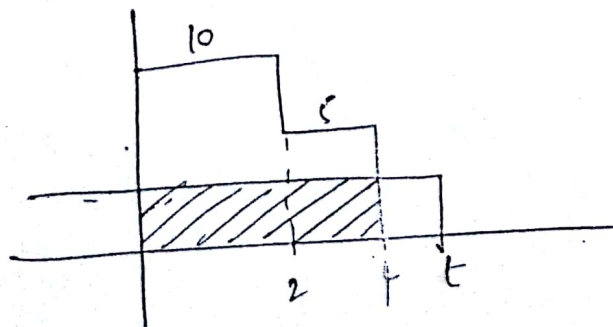


$$y(t) = \int_0^2 10 \times 1 d\tau + \int_2^t 5 \times 1 d\tau$$

$$= 5(t+2)$$

Case (4):

$$t > 4$$



$$y(t) = \int_0^2 10 \times 1 d\tau + \int_2^4 5 \times 1 d\tau$$

$$= 30$$

Sun

23/2/17

Soln

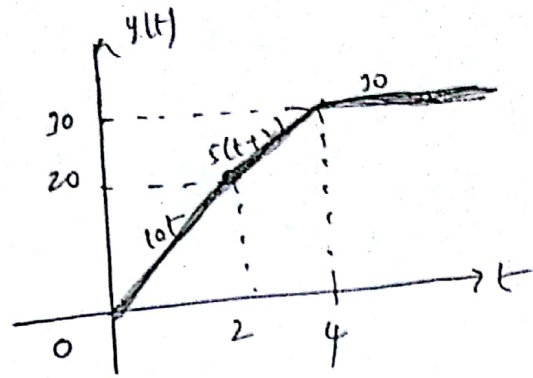
Summarise

$$y(t) = 0; t < 0$$

$$= 10t; 0 < t < 2$$

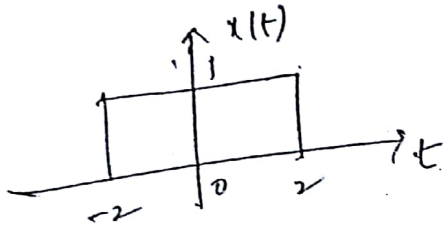
$$= 5(t+2); 2 < t < 4$$

$$= 30; t > 4$$

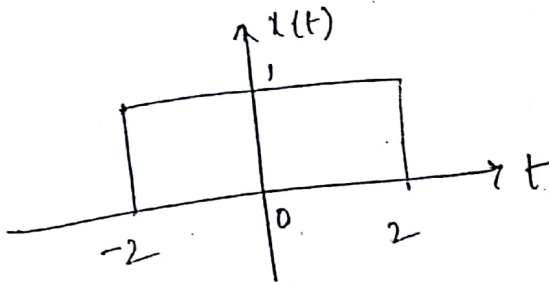


23/2/13

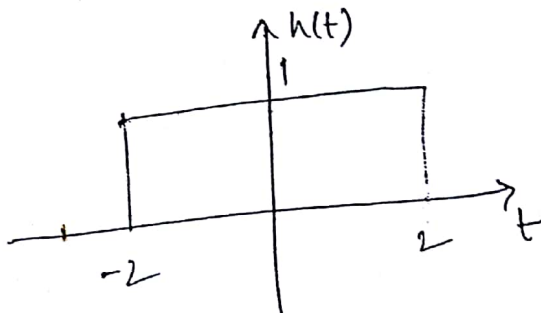
* Find the o/p response when the signal is convolved with itself-



Sol:-



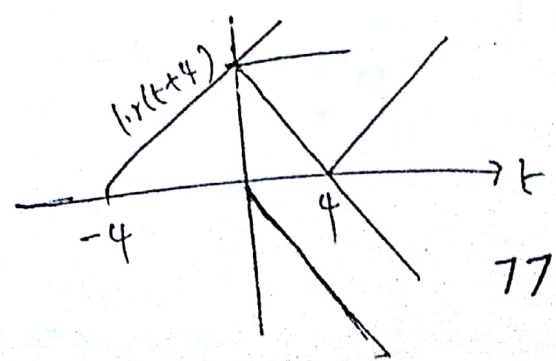
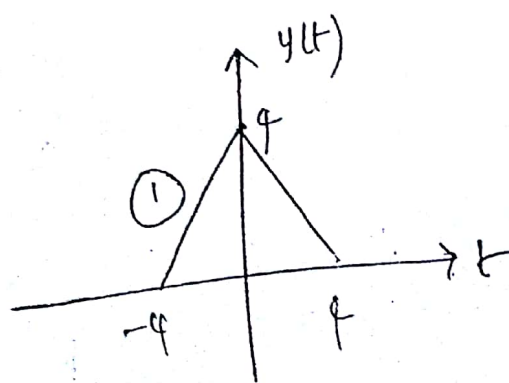
$$\Rightarrow [u(t+2) - u(t-2)]$$



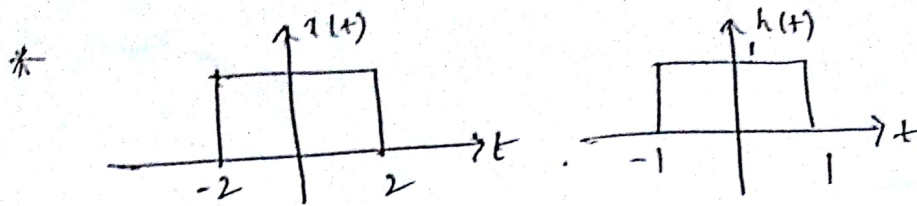
$$u[(t+2) - u(t-2)]$$

$$= [r[(t+2+2) - r(t+2-2) - r(t+2-2) + r(t-2-2)]]$$

$$= r(t+4) - r(t) - r(t) + r(t-4)$$



NOTE: The convolution of two rectangular pulses of equal width produces a triangle.



Sol:-

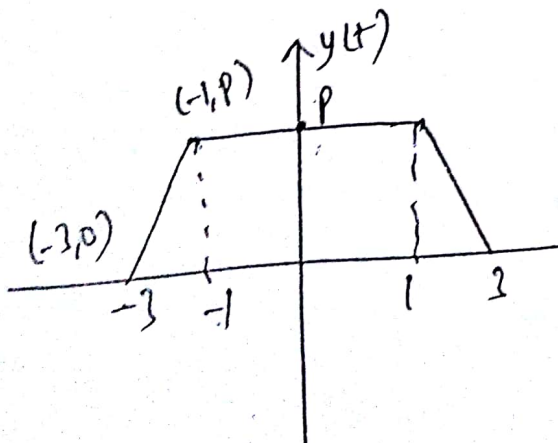
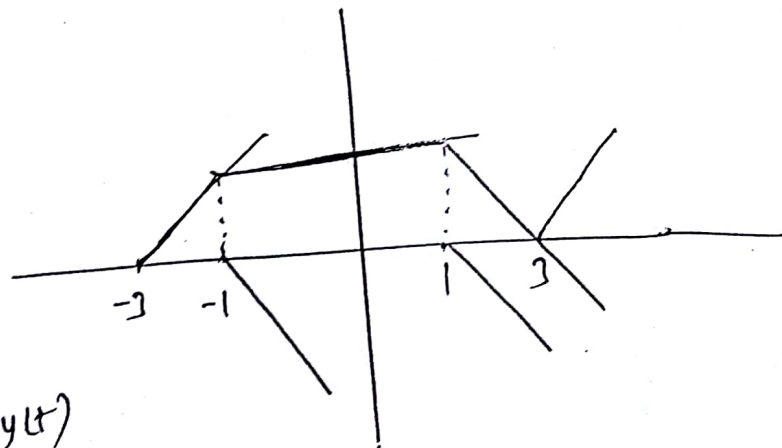
$$x(t) = [u(t+2) - u(t-2)]$$

$$h(t) = [u(t+1) - u(t-1)]$$

$$= x(t+2+1) - x(t+2-1) - x(t-2+1) + x(t-2-1)$$

$$= x(t+3) - x(t+1) - x(t-1) + x(t-3)$$

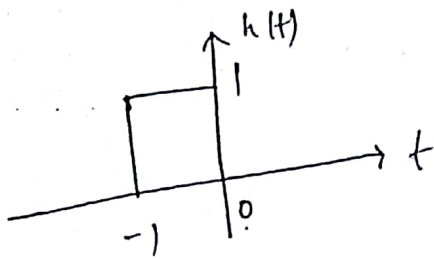
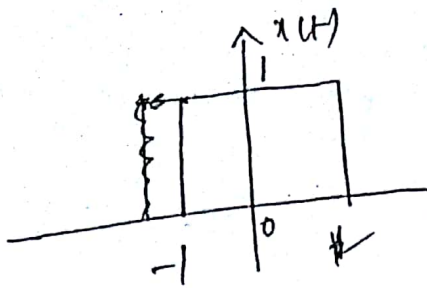
$$= x(t+3) - x(t+1) - x(t-1) + x(t-3)$$



$$\frac{p}{-1+1} = 1$$

$$p = 2$$

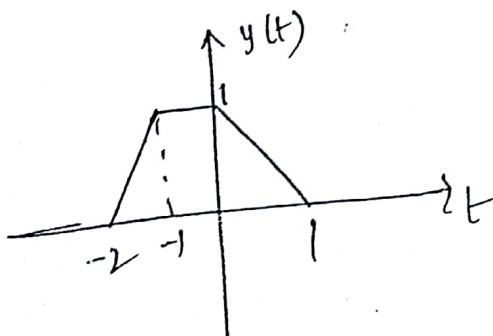
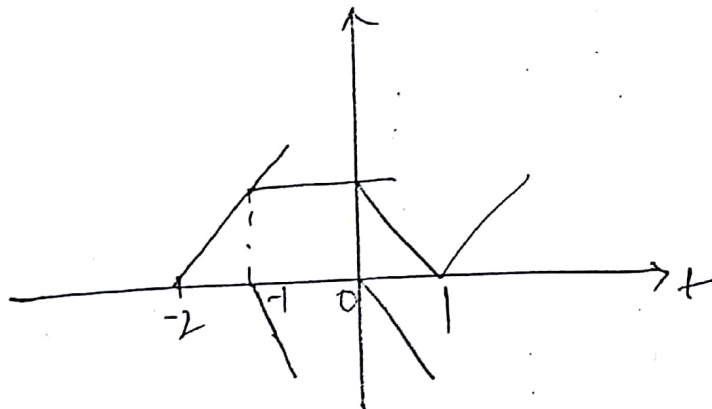
② → The convolution of two rectangular pulse of unequal lengths produces a trapezium.



$$x(t) = [u(t+1) - u(t-1)]$$

$$h(t) = [u(t+1)] - u(t)]$$

$$x(t) * h(t) = h(t+2) - h(t) - h(t+1) + h(t-1)$$



Discrete Convolution

→ circular conv. (periodic seq.)

$$y(n) = x(n) * h(n)$$

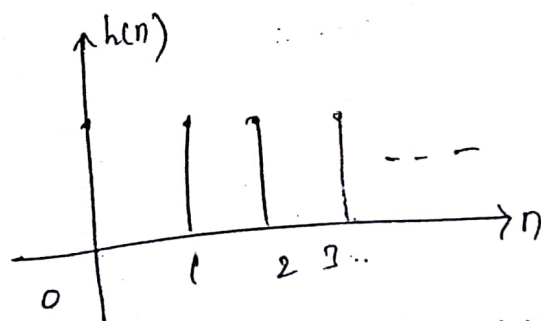
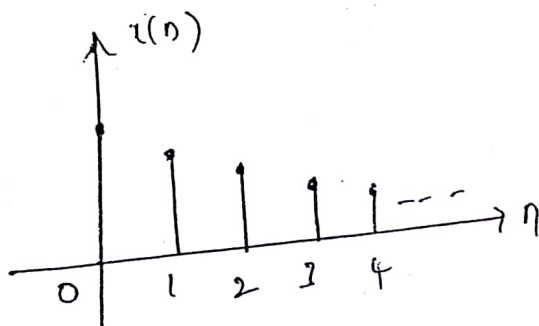
$$= \sum_{k=-\infty}^{\infty} x(k) h(n-k) \text{ or } \sum_{k=-\infty}^{\infty} x(n-k) h(k).$$

* Find the o/p response when the signal $x(n] = a^n u(n)$ where $0 < a < 1$ convolved with $h(n) = u(n)$.

Sol:-

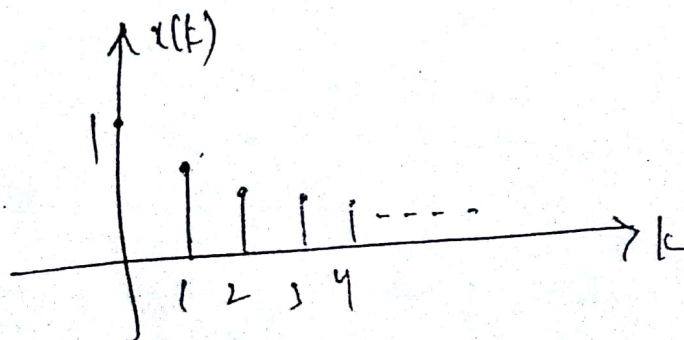
$$x(n) = a^n u(n); 0 < a < 1$$

$$h(n) = u(n)$$

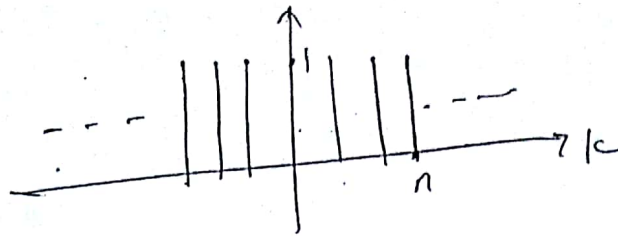


Limits of o/p response

$$0 \leq n < \infty$$

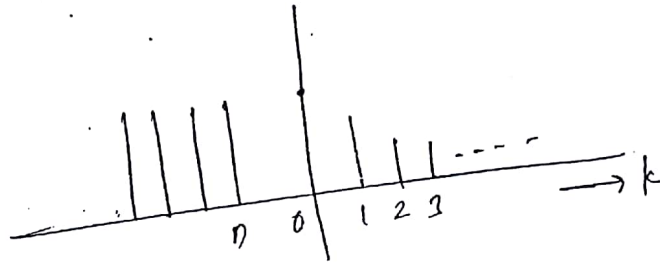


$$h(n-k) = u(n-k)$$



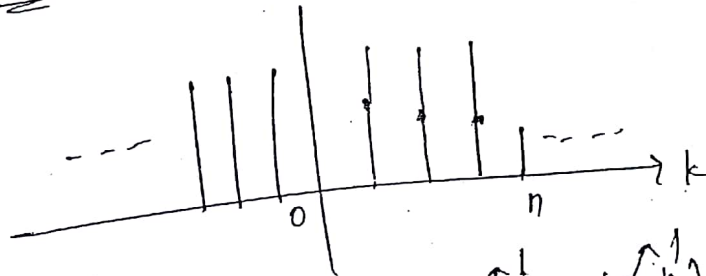
where

Case 1: $n < 0$



$$y(n) = 0;$$

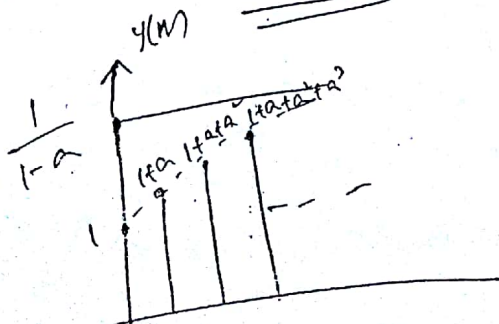
Case 2: $n \geq 0$



$$y(n) = \sum_{k=0}^n a^k y(k) u(n-k)$$

$$= \frac{1-a^{n+1}}{1-a}; \quad n \geq 0$$

Summarise



$$y(n) = 0; \quad n < 0$$

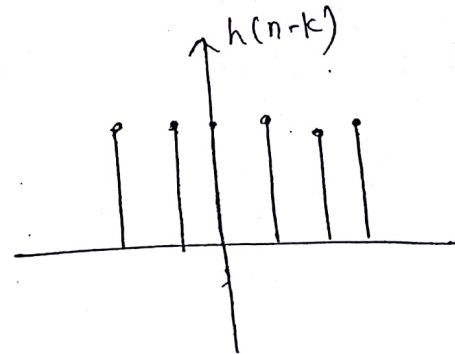
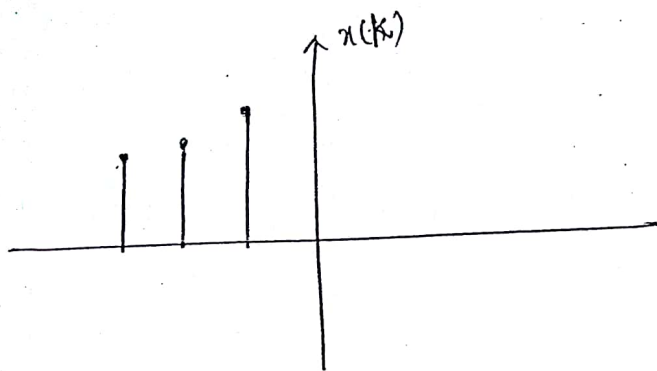
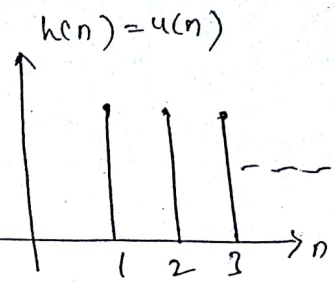
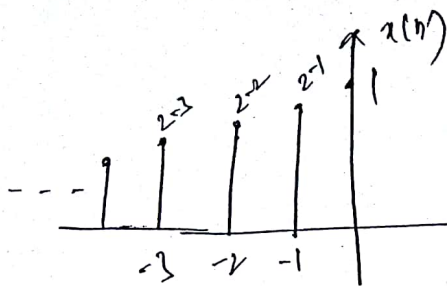
$$= \frac{1-a^{n+1}}{1-a}; \quad n \geq 0$$

Find the o/p response when $x(n) = a \cdot u(-n)$, convolved with $h(n) = u(n)$.

Sol:-

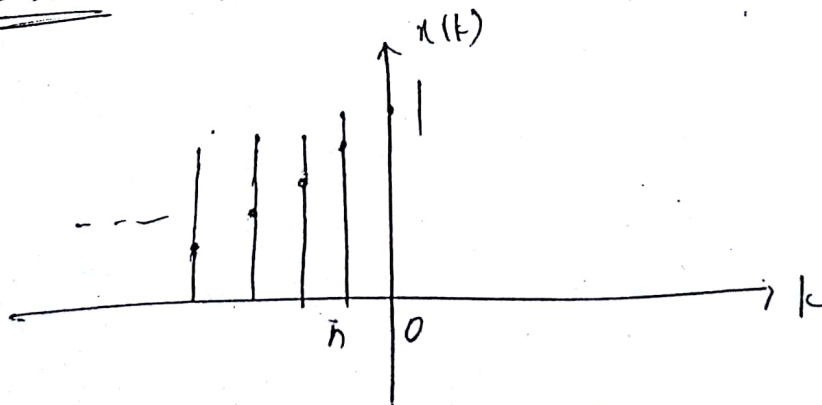
$$x(n) = a^n u(-n)$$

$$; \quad -\infty < n < \infty$$



Case (i):

$$n < 0$$



$$y(n) = \sum_{k=-\infty}^n a^k u(k) u(n-k)$$

$$= \sum_{k=-\infty}^n a^k ; \quad \text{let } l = -k$$

$$= \sum_{l=-n}^{\infty} \left(\frac{1}{2}\right)^l$$

$$\text{Let } m = 1+n$$

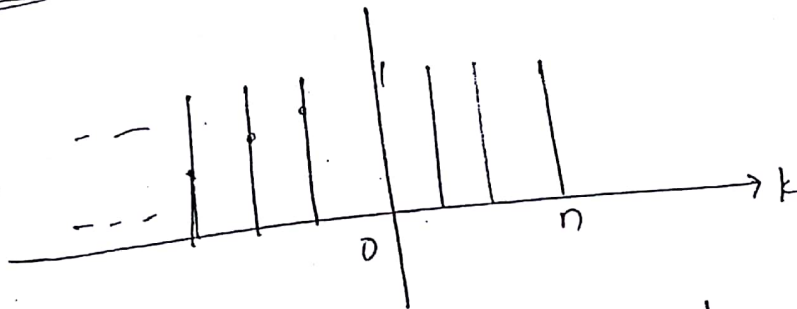
$$= \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{m-n}$$

$$= \left(\frac{1}{2}\right)^{-n} \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m$$

$$= 2^n \times \frac{1}{1 - \frac{1}{2}}$$

$$= 2^{n+1}; n < 0$$

Case (2) $n \geq 0$

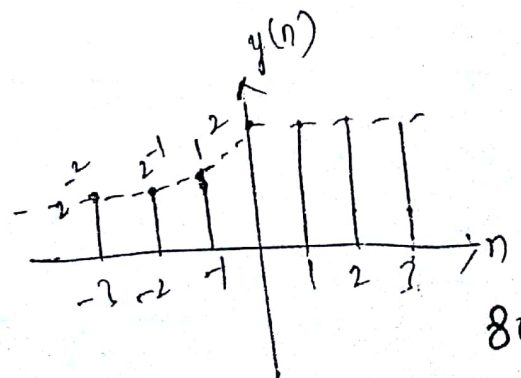


$$y(n) = \sum_{k=-\infty}^{\infty} 2^k u(k) u(n-k)$$

$$= \sum_{k=0}^{\infty} 2^{-k}$$

$$= \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$$

$$= \frac{1}{1 - \frac{1}{2}} = 2$$



Summarise

$$y(n) = 2^{n+1}; n < 0$$

$$= 2; n \geq 0.$$

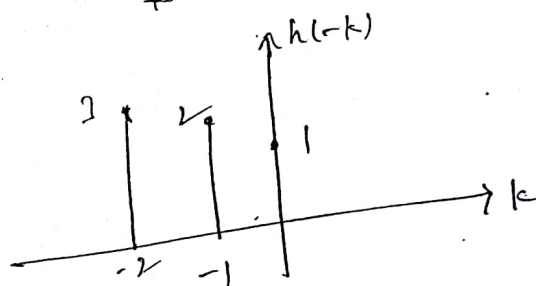
* Find the o/p response when $x(n) = \{3, 2, 0, 1\}$ convolved with $h(n) = \{1, 2, 3\}$

Soln

Limit of o/p response

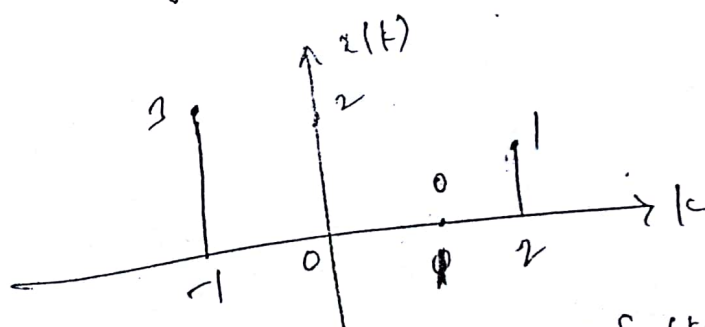
$$-1+0 \leq n \leq 2+2$$

$$\boxed{-1 \leq n \leq 4}$$

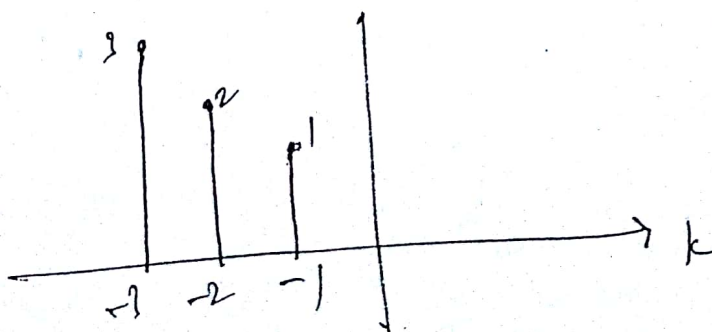


$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

$$y(-1) = \sum x(k) h(-1-k)$$

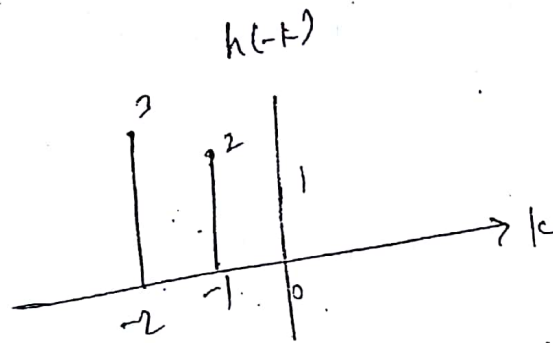
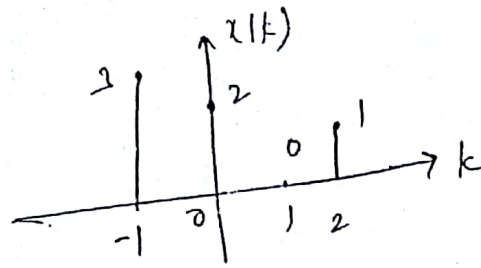


$$h(-1-k) = h[-(k+1)]$$



$$y(-1) = 3 \times 1 = 3$$

$$y(n) = \sum x(k) h(n-k)$$



$$y(n) = (3 \times 2) + (2 \times 1) = 6 + 2 = 8$$

Tabular method for linear convolution

n	-3	-2	-1	0	1	2	3	4
x(k)			3	2	0	1		
h(k)				1	2	3		
h(-k)		3	2	1				
h(-1-k)	3	2	1					
h(-k)		3	2	1				
h(1-k)			3	2	1			
h(2-k)				3	2	1		
h(3-k)					3	2	1	
h(4-k)						3	2	1

$$y(-1) = 3 \times 1 = 3$$

$$y(0) = 3 \times 2 + 2 \times 1 = 8$$

$$y(1) = 3 \times 2 + 2 \times 2 + 0 \times 1 = 10$$

$$y(2) = 6 + 0 + 1 = 7$$

$$y(3) = 0 + (1 \times 2) = 2$$

$$y(4) = 1 \times 3 = 3$$

$$y(n) = \{ \underset{\uparrow}{3}, 8, 3, 7, 2, 3 \}$$

find the o/p response when $x(n) = 2\delta(n+2) - \delta(n-1)$ convolved with $h(n) = 3\delta(n) + \delta(n-1) + \delta(n-2)$

soln-

$$x(n) = 2\delta(n+2) - \delta(n-1)$$

$$x(n) = \{ \underset{\uparrow}{2}, 0, 0, -1 \}$$

$$h(n) = \{ \underset{\uparrow}{3}, 1, 1 \}$$

limits $\Rightarrow -2 \leq n \leq 3$

n	-4	-3	-2	-1	0	1	2	3
$x(k)$			2	0	0	-1		
$h(k)$					3	1	1	
$h(-k)$			1	1	3			
$h(-1-k)$		1	1	3				
$h(-k)$			1	1	3			
$h(1-k)$				1	1	3		
$h(2-k)$					1	1	3	
$h(3-k)$						1	1	3

$$y(n) = \{ 6, 2, 2, -3, -1, -1 \}$$

↑

Linear convolution by diagonal method:

	2	0	0	-1
2	6	0	0	-3
1	2	0	0	-1
1	2	0	0	-1

$$y(n) = \{ 6, 2, 2, -3, -1, -1 \}$$

↑

4/3/13

Correlation:

Convolution

- (1) The output to a system response is obtained by the process of convolution

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \text{ (or)} \\ = \int_{-\infty}^{\infty} x(t-\tau) h(\tau) d\tau$$

Correlation

- (1) It measures the similarity between two signals.

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t) y(t-\tau) dt$$

Sol:-

Classification

- (1) Autocorrelation (ACF): ACF measures the similarity b/w the signal and delayed version of the same signal given as

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t) x(t-\tau) dt \text{ (or)} \\ = \int_{-\infty}^{\infty} x(t+\tau) x^*(t) dt$$

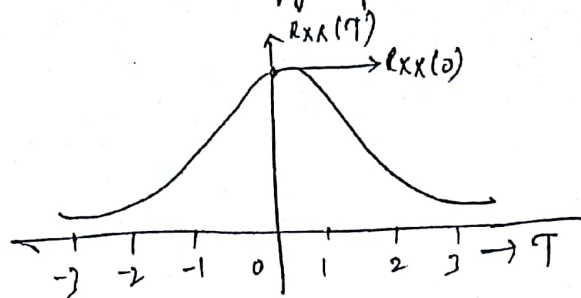
- (2) Cross correlation (CCF): It measures the similarity b/w the signal and delayed version of the other signal given

$$\text{as } R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t) y(t-\tau) dt \text{ (or)}$$

$$R_{yx}(\tau) = \int_{-\infty}^{\infty} y(t) x(t-\tau) dt$$

Properties of ACF:

- (1) ACF is purely function of T .
- (2) At $T=0$ ACF = energy of the signal.
- (3)



At $T=0$ the correlation function ACF will have the max. value and $R_{xx}(T)$ is always less than or equal to $R_{xx}(0)$ i.e.

$$R_{xx}(T) \leq R_{xx}(0)$$

- (4) $\psi(\omega) \rightarrow$ E.S.D. The fourier transform of ACF produces the energy spectral density. So, therefore ESD and ACF forms fourier transform pair.

$$\underbrace{\psi(\omega)}_{\text{E.S.D}} = \int_{-\infty}^{\infty} R_{xx}(T) e^{-j\omega T} dT$$

$$\boxed{\psi(\omega) \longleftrightarrow R_{xx}(T)} \rightarrow \text{F.T pair.}$$

- (5) ACF of periodic signals is given as

$$R_{xx}(T) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) x(t-T) dT$$

At $T=0$ the ACF = power of the signal $x(t)$.⁽⁶⁾ The fourier transform of the periodic ACF produces the power spectral density given as

Problem

- (1) Find the auto correlation func of the signal $x(t) = e^{-at}u(t)$ and also find the conditions in which the ACF is equal to energy of the signal $x(t)$,

Sol:-

$$x(t) = e^{-at}u(t); a > 0$$

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)x(t-\tau) dt$$
$$= \int_{-\infty}^{\infty} e^{-at}u(t) e^{-a(t-\tau)}u(t-\tau) dt.$$

$$= \int_{-\infty}^{\infty} e^{-at}e^{-a(t-\tau)}e^{a\tau} dt$$

$$= \int_{-\infty}^{\infty} e^{-2at+a\tau} dt$$

$$= \frac{e^{a(-2t+\tau)}}{-2a} \Big|_{-\infty}^{\infty}$$

$$= \frac{e^{-2a\tau}}{-2a} \Big|_{-\infty}^{\infty} = \frac{e^{-a\tau}}{2a}$$

$$\text{If } \tau = 0.$$

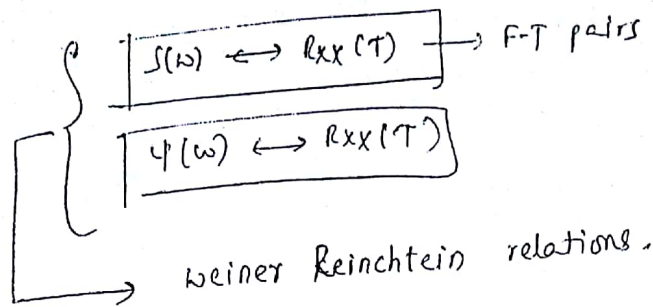
$$R_{xx}(0) = \frac{1}{2a}$$

$$x(t) = e^{-at}u(t)$$

$$\int_{-\infty}^{\infty} |e^{-at}u(t)|^2 dt$$

$$= \int_0^{\infty} e^{-2at} dt = \frac{e^{-2a}}{-2a} \Big|_0^{\infty} = \frac{1}{2a}$$

$$S(\omega) = \int_{-\infty}^{\infty} R_{xx}(T) e^{-j\omega T} dT ; R_{xx}(T) \rightarrow \text{ACF of periodic signals}$$



* Energy spectral Density (ESD)
 $\downarrow$
 $\Psi(\omega)$

(1) It gives the distribution of energy of a signal in its frequency domain.

(2) The area under ESD gives energy of the signal.

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi(\omega) d\omega = \int_{-\infty}^{\infty} \phi(t) dt$$

$$\boxed{\Psi(\omega) = |X(\omega)|^2}$$

3) $R_{xx}(T) \leftrightarrow \Psi(\omega)$

4) $E_{[x_1(t) + x_2(t)]} = E_1 + E_2$

if $x_1(t)$ & $x_2(t)$ are orthogonal fns.

Power spectral Density (PSD)
 $\downarrow$
 $S(\omega)$

(1) It gives the distribution of power of a signal in its frequency domain.

(2) The area under PSD gives power of a signal.

$$P = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) d\omega = \int_{-\infty}^{\infty} s(f) df$$

$$\boxed{S(\omega) = \lim_{T \rightarrow \infty} \frac{|X(\omega)|^2}{T}}$$

$S(\omega) \neq$

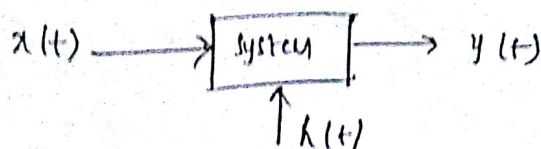
3) $R(T) \leftrightarrow S(\omega)$

4) $P_{(x_1(t) + x_2(t))} = P_1 + P_2$

5) $S_y(\omega) = |H(\omega)|^2 S_x(\omega)$
 in & out power

* Relation b/w the input and output energies of a LTI systems.

ESD:



$$y(t) = x(t) * h(t)$$

$$Y(\omega) = X(\omega) H(\omega)$$

$$\Psi_Y(\omega) = \Psi_X(\omega) \times \Psi_H(\omega)$$

$$\boxed{\Psi_Y(\omega) = |H(\omega)|^2 \Psi_X(\omega)}$$

$\Psi_X(\omega) \rightarrow$ Energy of i/p

$\Psi_Y(\omega) \rightarrow$ Energy of o/p.

PSD:

$$\boxed{S_Y(\omega) = |H(\omega)|^2 S_X(\omega)} \rightarrow \text{i/p and o/p power relations of a LTI systems.}$$

Problem:

(1) A filter has an input response $x(t) = e^{-2t}u(t)$ and impulse response $h(t) = e^{-3t}u(t)$. Find the ESD of the o/p response

Sol:-

$$x(t) = e^{-2t}u(t)$$

$$h(t) = e^{-3t}u(t)$$

$$y(t) = x(t) * h(t)$$

$$= e^{-2t}u(t) * e^{-3t}u(t)$$

$$Y(\omega) = \frac{1}{1+j\omega} \times \frac{1}{3+j\omega}$$

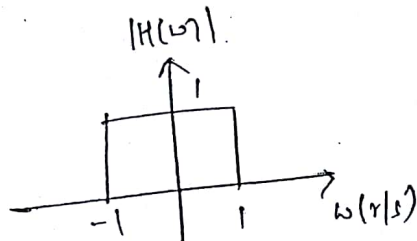
$$\Psi_Y(\omega) = |H(\omega)|^2 |X(\omega)|^2 = \left(\frac{1}{1+\omega^2} \right) \times \left(\frac{1}{9+\omega^2} \right)$$

(2) A signal $x(t) = e^{-2t} u(t)$ is passed to an ideal low pass filter with cut-off frequency of 1 rad/sec then find the i/p and o/p energies of the signal

Soln

$$x(t) = e^{-2t} u(t)$$

$$= |X(\omega)|^2 = \left(\frac{1}{2+j\omega} \right)^2 = \frac{1}{4+\omega^2}$$



$$\psi_x(\omega) = |H(\omega)|^2 \psi_x(\omega)$$

$$E \psi_x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi_x(\omega) d\omega ; |\omega| < 1$$

$$= \frac{1}{2\pi} \int_{-1}^1 |X(\omega)|^2 d\omega$$

$$= \int_{-1}^1 \frac{1}{4+\omega^2} d\omega$$

$$= \left[\frac{1}{2} \tan^{-1}\left(\frac{\omega}{2}\right) \right]_{-1}^1$$

$$= \frac{1}{2} \left(\tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}\left(-\frac{1}{2}\right) \right)$$

$$= \tan^{-1}\left(\frac{1}{2}\right) = \frac{26.56}{2\pi} = 0.0735$$

(3) For the signal $x(t) = e^{-at} u(t)$. Find out the total energy contained in the frequency band $|f| \leq f_1$, where $f_1 = \frac{a}{2\pi}$.

Sol:-

$$x(t) = e^{-at} u(t)$$

$$X(\omega) = \frac{1}{a+j\omega}$$

$$|X(\omega)|^2 = \frac{1}{a^2 + \omega^2}$$

$$E_{\psi}(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi_X(w) dw$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{a^2 + \omega^2} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{a^2 + \omega^2} d\omega$$

$$= \frac{1}{2\pi} \left[\tan^{-1}\left(\frac{\omega}{a}\right) - \tan^{-1}\left(\frac{\omega}{a}\right) \right]_{-\infty}^{\infty}$$

$$= \frac{1}{2\pi a}$$

Sol:-

Q. Find the ACF of the signal $x(t) = A \sin(\omega_0 t + \phi)$ & also calculate the average power of the correlated signal

Sol:-

$$R_{xx}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) x(t-\tau) dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T A \sin(\omega_0 t + \phi) A \sin(\omega_0 (t-\tau) + \phi) dt$$

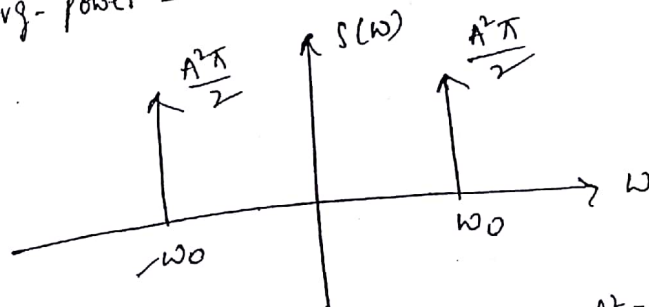
$$= \lim_{T \rightarrow \infty} \frac{A^2}{2T} \int_{-T}^T 2 \sin(\omega_0 t + \phi) \sin(\omega_0 (t-\tau) + \phi) dt$$

$$\begin{aligned}
 &= \lim_{T \rightarrow \infty} \frac{1}{T} \frac{A^2}{4T} \int_{-T}^T \cos(\omega_0 t + \phi + \omega_0 t - \omega_0 T) - \cos(\omega_0 t + \phi - \omega_0 t + \omega_0 T) dt \\
 &= \lim_{T \rightarrow \infty} \frac{1}{T} \frac{A^2}{4T} \int_{-T}^T \cos(\omega_0 T) - \cos(2\omega_0 t - \omega_0 T + \phi) dt \\
 &= \lim_{T \rightarrow \infty} \frac{1}{T} \frac{A^2}{4T} \left[\frac{\sin(2\omega_0 t - \omega_0 T + \phi)}{2\omega_0} + \cos \omega_0 T (2T) \right]_{-T}^T \\
 &= \lim_{T \rightarrow \infty} \frac{1}{T} \cdot \frac{A^2}{4T} \times \cos \omega_0 T \times 2T = \frac{A^2}{2} \cos(\omega_0 T)
 \end{aligned}$$

P.S.D $S(\omega) = \mathbb{E}[R_{XX}(T)]$

$$S(\omega) = \frac{A^2}{2} [\pi \delta(\omega + \omega_0) + \pi \delta(\omega - \omega_0)]$$

Avg. power = Area under the P.S.D



$$\text{Avg. power} = \frac{A^2 \pi}{2} + \frac{A^2 \pi}{2} = \frac{A^2 \pi}{1}$$

Cross Correlation (C.C.F.):

Properties:

(1) $R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t) x_2(t-\tau) dt$

$$R_{21}(\tau) = \int_{-\infty}^{\infty} x_2(t) x_1(t-\tau) d\tau$$

$$R_{12}(\tau) \neq R_{21}(\tau)$$

$$(2) R_{12}(\tau) = R_{21}^*(-\tau)$$

$$(3) R_{12}(\tau) = X_1(\omega) \times X_2^*(-\omega)$$

(4) If $x_1(t)$ & the energy signals $x_1(t)$ and $x_2(t)$ are orthogonal over the entire interval then $R_{12}(0) = 0$ i.e. if

$$\int_{-\infty}^{\infty} x_1(t) x_2^*(t) dt = 0$$

$$(\psi_1(\tau) + \psi_2(\tau))^* \neq \psi_1^*(\tau) + \psi_2^*(\tau) + 2\psi_1(\tau)\psi_2(\tau)$$

$$(5) (\psi_1(\tau) + \psi_2(\tau))^* = \psi_1^*(\tau) + \psi_2^*(\tau)$$

if signals $x_1(t)$ & $x_2(t)$ are orthogonal to each other

$\psi_1(\tau) \rightarrow$ E.S.D of signal $x_1(t)$.

$\psi_2(\tau) \rightarrow$ E.S.D of signal $x_2(t)$

Problems

* find the ccf of the signals $x_1(t) = e^{-t}u(t)$ and $x_2(t) = e^{-3t}u(t)$

Sol:-

$$x_1(t) = e^{-t}u(t)$$

$$x_2(t) = e^{-3t}u(t)$$

$$R_{12}(\tau) = \int_{-\infty}^{\infty} e^{-t}u(t) e^{-3(t-\tau)}u(t-\tau) dt$$

$$= \int_{-\infty}^{\infty} e^{-t} e^{-3t} e^{3\tau} dt$$

$$= \int_T^{\infty} e^{-4t} e^{3\tau} dt \cdot e^{3\tau} \left[\frac{e^{-4t}}{-4} \right]_T^{\infty}$$

$$= \frac{e^{3\tau}}{-4} \left[0 - e^{-4T} \right] = \frac{e^{-T}}{4}$$

* fir

Sol:

$$R_{12}(T) = \frac{e^{-T}}{4}$$

$$R_{21}^*(T) = \int_{-\infty}^{\infty} x_2(t) x_1(t+T) dt.$$

$$= \int_{-\infty}^{\infty} e^{-2t} u(t) e^{-(t+T)} u(t+T) dt$$

$$= \int_{-\infty}^{\infty} e^{-2t} e^{-t} e^{-T} dt$$

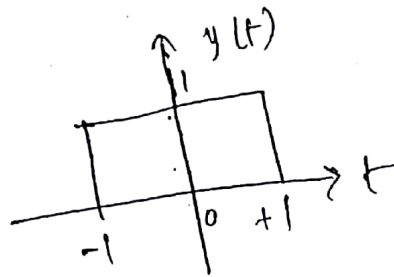
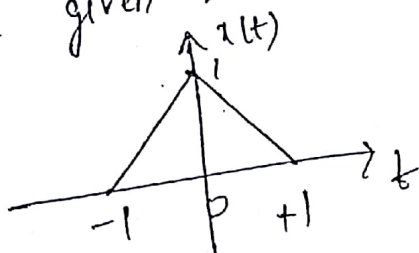
$$= e^{-T} \left[\frac{e^{-4t}}{-4} \right]_0^{\infty}$$

$$= \frac{e^{-T}}{4}$$

$$\therefore R_{12}(T) = R_{21}^*(-T)$$

* find the CCF b/w the given triangular function and gate function given below.

Sol:

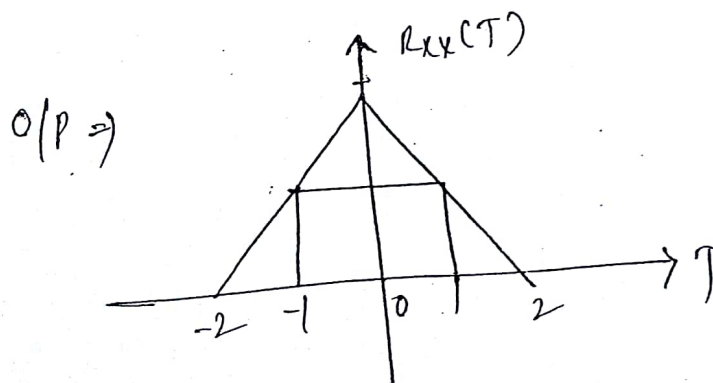
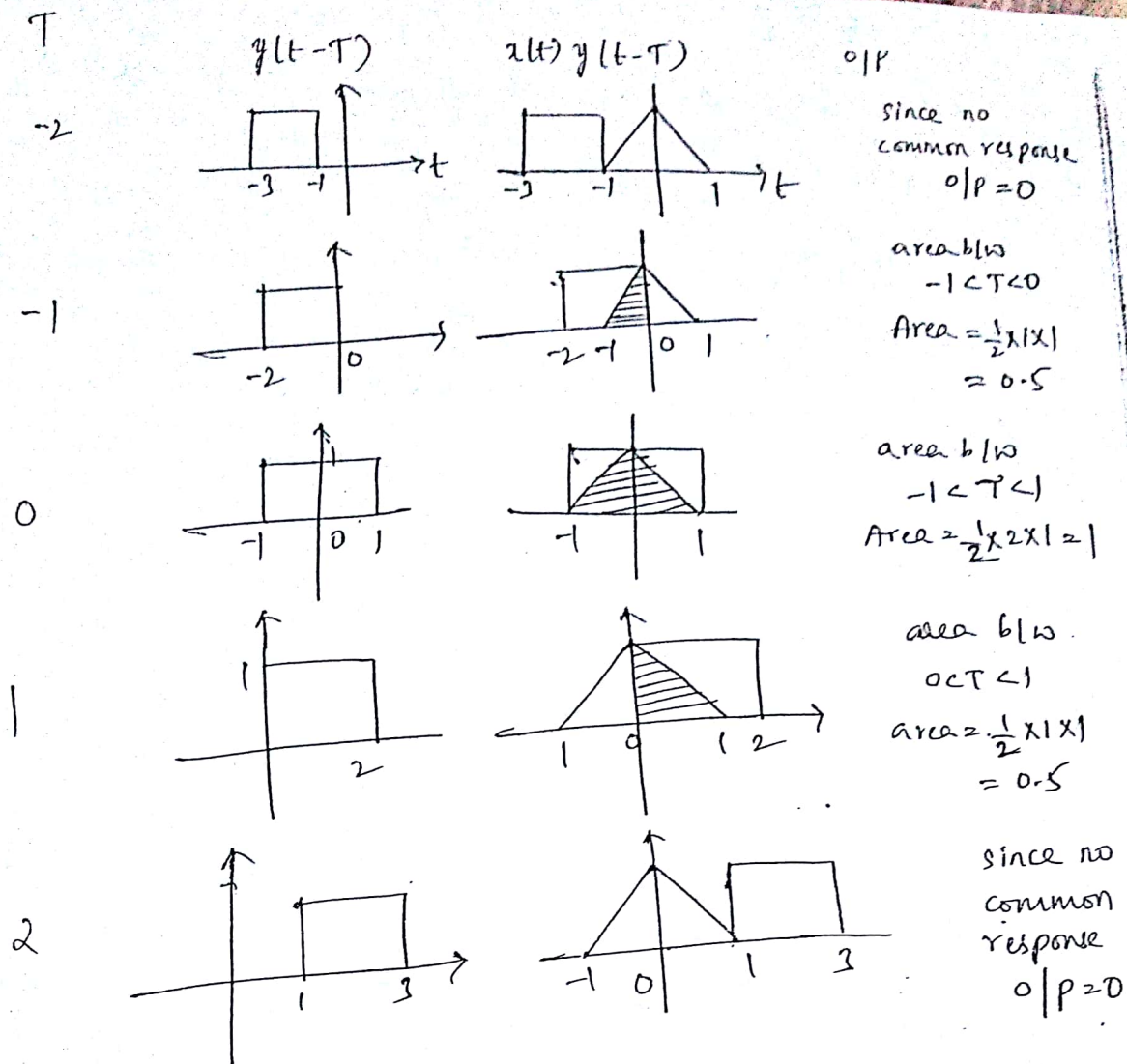


(1) Finding limits of o/p correlation using limits rule

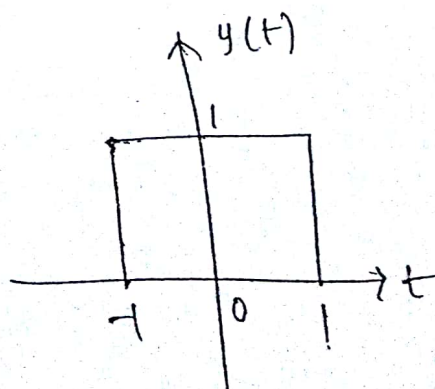
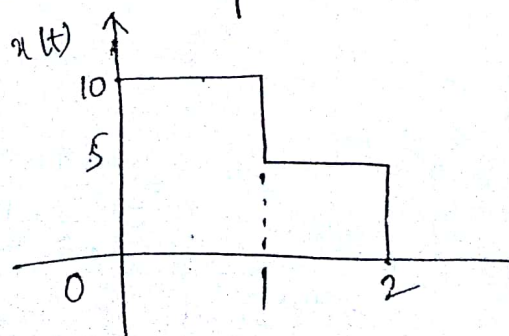
$$-1-1 < T < 1+1$$

$$\boxed{-2 < T < 2}$$

$$R_{xy}(T) = \int_{-\infty}^{\infty} x(t) y(t-T) dt$$



* find the CCF of the signals



$$-1 < T < 3$$

T

$y(t-T)$

$x(t) y(t-T)$

no common
response
o/p = 0

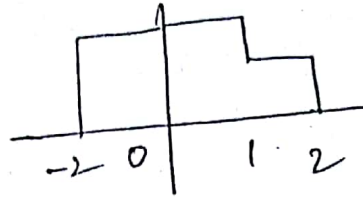
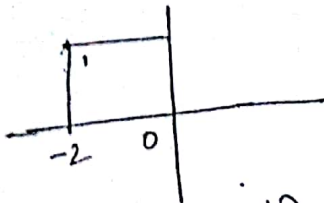
area b/w
 $0 < T < 1$
Area = $\frac{6}{8} \times 1 \times 5$
= 5

area b/w
 $1 < T < 2$
Area = $\frac{6}{8} \times 2 \times 5$
= 10

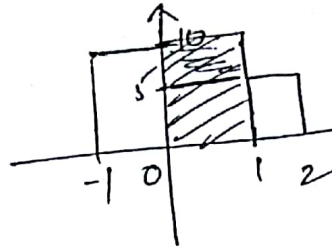
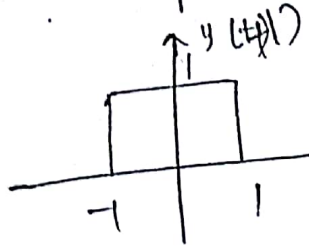
$A = 1 \times 5$
= 5 units

$A = 0$

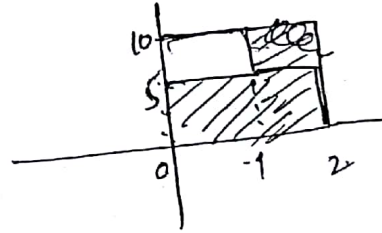
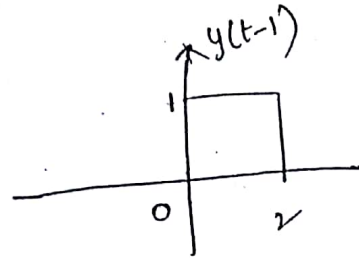
-1



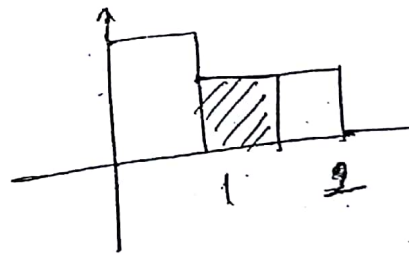
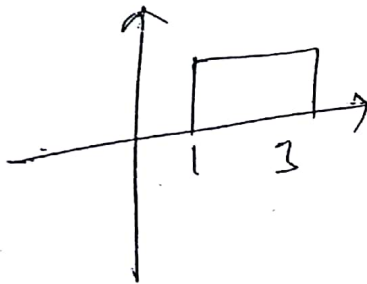
0



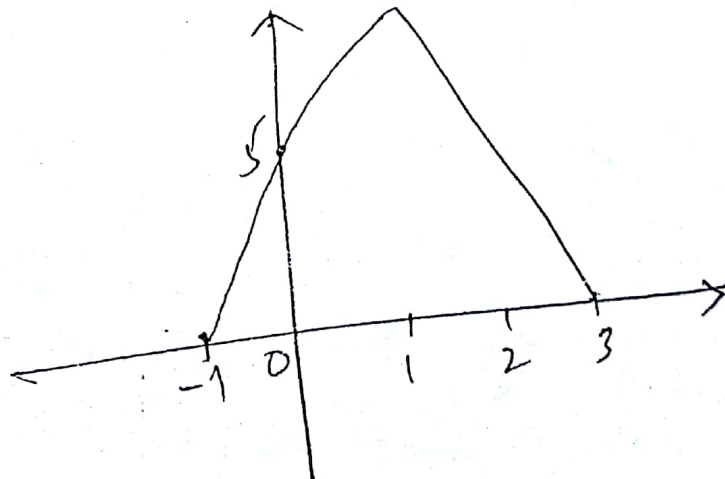
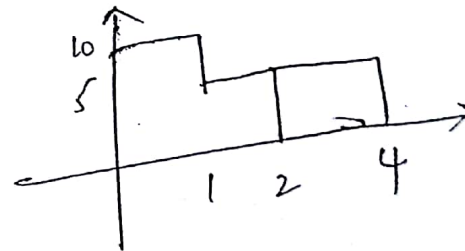
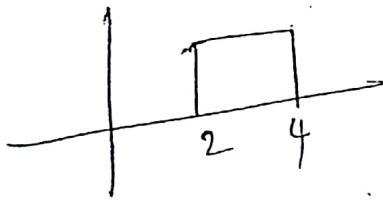
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2



3.



Discrete Correlation:

Auto correlation (ACF)

Cross correlation (CCF)

$$ACF \quad r_{xx}(k) = \sum_{n=-\infty}^{\infty} x(n)x(n-k)$$

$$r_{xy}(k) = \sum_{n=-\infty}^{\infty} x(n)y(n-k)$$

* Mathematically convolution and correlation are almost same only the difference is -

$$\begin{aligned} y(n) &= x_1(n) * x_2(n) \rightarrow \text{convolution} \\ R_{12}(k) &= x_1(n) * x_2(-n) \rightarrow \text{correlation function} \end{aligned}$$

→ $R_{11}(k) = x_1(n) * x_1(-n) \rightarrow ACF$ is always an even function

(1) ACF $R_{11}(k) = R_{11}^*(-k)$

CCF $R_{12}(k) = R_{21}^*(-k)$

(2) ACF $R_{11}(k) \leq R_{11}(0)$

CCF $E R_{12}(k) \leq \sqrt{E_1 E_2}$

where E_1, E_2 are energies of signals $x_1(n)$ & $x_2(n)$

Energy of CCF

③ using maximum value of CCF the normalised CCF is defined as $P_{xy}(m) = \frac{r_{xy}(m)}{\sqrt{r_{xx}(0)r_{yy}(0)}}$
for ACF $P_{xx}(m) = \frac{r_{xx}(m)}{r_{xx}(0)}$

Problem

* find the CCF of the signals $x_1(n) = \{-2, 1, 0, 3\}$ and $x_2(n) = \{1, 2, 3\}$

Sol:-

$$x_1(n) = \{-2, 1, 0, 3\}$$

$$x_2(n) = \{1, 2, 3\}$$

$$R_{12}(k) = x_1(n) * x_2(-n) \rightarrow x_2(-n) = \{3, 2, 1\}$$

By Diagonal method

	-2	1	0	3
3	-6	3	0	9
2	-4	2	0	6
1	-2	1	0	3

$$R_{12}(k) = \{-6, -1, 0, 10, 6, 3\}$$

Limits Rule:

$$-1-2 \leq k \leq 2+0$$

$$-3 \leq k \leq 2$$

* find the ACF of the signal $x(n) = \{1, 2, 3\}$

Sol:-

$$x(n) = \{1, 2, 3\}$$

$$x(-n) = \{3, 2, 1\}$$

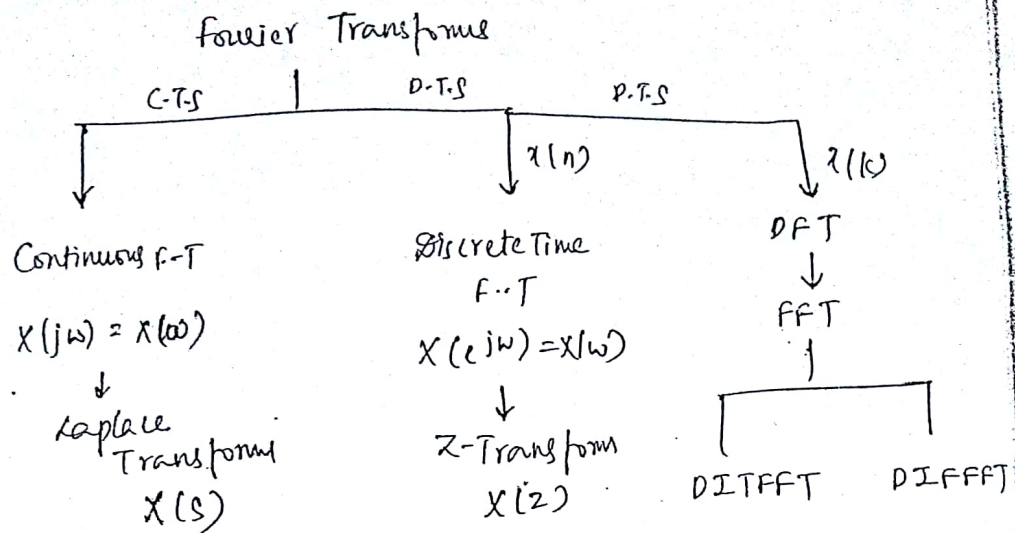
	1	2	3
3	3	6	9
2	2	4	6
1	1	2	3

$$-2 \leq k \leq 2$$

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$$R_{11}(k) = \{3, 8, 14, 8, 3\}$$

* Frequency domain representation of discrete signals



Note

* Jk

* S

(1) Find

Sol:-

* Discrete Time F.T

→ The frequency domain representation of discrete time signals is given as $X(j\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$.

$$X(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n}$$

Inverse D.T.F.T: $x(n)$

$$\mathcal{F}^{-1} [X(j\omega)] = x(n)$$

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(j\omega) e^{j\omega n} d\omega$$

* Conditions or Convergence of D.T.F.T

Dirichlet conditions:

Necessary conditions

(1) The discrete signal $x(n)$ must be non-periodic

c2) The signal $x(n]$ must be absolutely summable.

$$\sum_{n=-\infty}^{\infty} x(n) < \infty$$

NOTE

* The spectrum of D.T.F.T is also a continuous frequency response.

* It has symmetrical magnitude response and anti-symmetrical phase response

(i) Find the fourier transform of the signal $x(n) = a^n u(n)$ and also sketch its magnitude and phase response

$$x(n) = a^n u(n); 0 < a < 1$$

$$\text{Sol: } x(n) = \sum_{n=-\infty}^{\infty} a^n u(n) e^{j\omega n}$$

$$= \sum_{n=0}^{\infty} [ae^{-j\omega}]^n$$

$$= \sum_{n=0}^{\infty} [ae^{-j\omega}]^n$$

$$X(j\omega) = \frac{1}{1 - ae^{-j\omega}}$$

$$|X(\omega)| = \sqrt{X(\omega) X^*(\omega)}$$

$$= \sqrt{\frac{1}{[1 - ae^{-j\omega}]} \times \frac{1}{[1 - ae^{j\omega}]}}$$

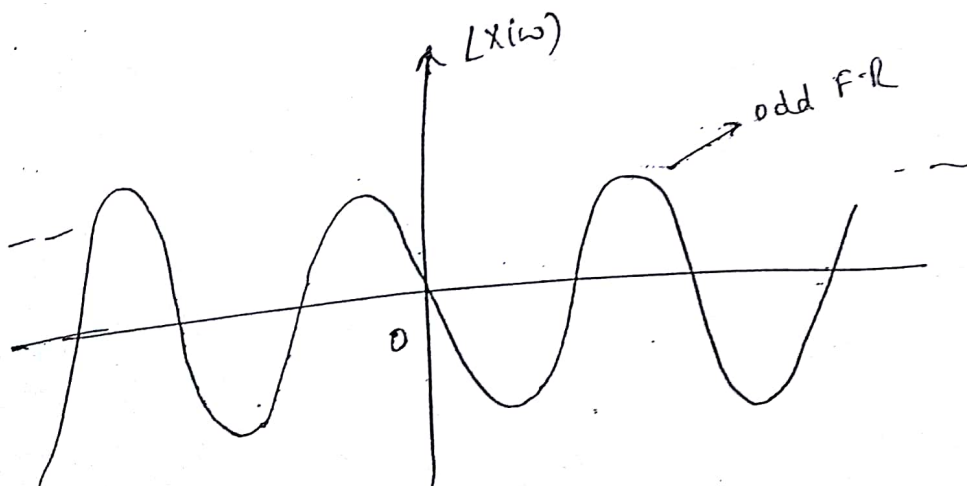
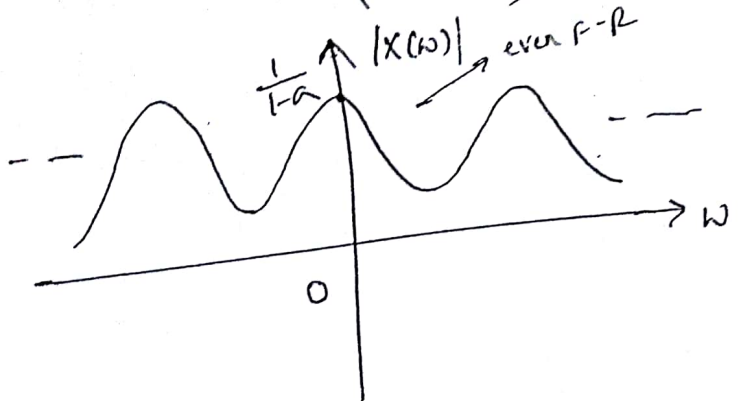
$$= \sqrt{\frac{1}{ae^{-j\omega} + a^2 e^{j\omega} + 1 - ae^{-j\omega}}}$$

$$|X(\omega)| = \frac{1}{\sqrt{1 - 2a \cos \omega + a^2}}$$

$$X(\omega) = \frac{1+j0}{[(1-a\cos\omega) + j(a\sin\omega)]}$$

$$\angle X(\omega) = \tan^{-1}\left(\frac{0}{1}\right) - \tan^{-1}\left(\frac{a\sin\omega}{1-a\cos\omega}\right)$$

$$= -\tan^{-1}\left(\frac{a\sin\omega}{1-a\cos\omega}\right)$$



Q2) Find 'N' point D.T.F.T of the sequence $x(n) = a^n$

$$x(n) = a^n$$

$$X(j\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} a^n e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} [ae^{-j\omega}]^n$$

$$X(j\omega) = \frac{1 - (ae^{-j\omega})^{N+1}}{1 - ae^{-j\omega}}$$

$$X(j\omega) = \frac{1 - a^{N+1}e^{-j\omega(N+1)}}{1 - ae^{-j\omega}}$$

(3) Find the D.T.F.T of $x(n) = \{1, 2, 0, 2, 1\}$

Sol:

$$X(\omega) = \sum_{n=-2}^2 x(n) e^{-j\omega n}$$

$$= x(-2)e^{j2\omega} + x(-1)e^{j\omega} + 0 + x(1)e^{-j\omega} + x(2)e^{-j2\omega}$$

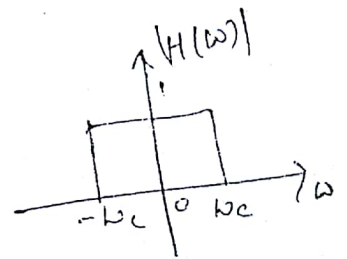
$$= 2 \left[\frac{e^{j2\omega} + e^{-j2\omega}}{2} \right] + 2 \left[\frac{e^{j\omega} + e^{-j\omega}}{2} \right]$$

$$= 6 \cos 2\omega + 4 \cos \omega$$

(4) An ideal low pass filter has a freq. response $H(\omega) = 1$ $|\omega| \leq \omega_c$. Find the impulse response of the ideal low pass filter.

Sol:

$$x(n) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} X(j\omega) e^{j\omega n} d\omega$$



$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} 1 \cdot e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\omega n}}{jn} \right]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{\pi n} \left[\frac{e^{j\omega_c n}}{jn} - \frac{e^{-j\omega_c n}}{jn} \right]$$

$$= \frac{\sin \omega_c n}{\pi n}$$

(1) Linearity Property:

$$\text{If } x_1(n) \xleftrightarrow{\text{DTFT}} X_1(\omega)$$

$$x_2(n) \xleftrightarrow{\quad} X_2(\omega)$$

$$\text{then } ax_1(n) + bx_2(n) \longleftrightarrow aX_1(\omega) + bX_2(\omega)$$

(2) Time Scaling Property

$$\text{If } x(n) \longleftrightarrow X[e^{j\omega}] \text{ then}$$

$$x\left(\frac{n}{K}\right) \longleftrightarrow X[e^{j\omega K}]$$

(3) Conjugate Property

$$\text{If } x(n) \longleftrightarrow X(\omega) \text{ then}$$

$$x^*(n) \longleftrightarrow X^*(-\omega)$$

(4) Time Shifting Property

$$\text{If } x(n) \longleftrightarrow X(\omega) \text{ then}$$

$$x(n-n_0) \longleftrightarrow e^{-j\omega n_0} X(\omega)$$

(5) Frequency Shifting Property

$$\text{If } x(n) \longleftrightarrow X(\omega)$$

$$\text{then } e^{j\omega_0 n} x(n) \longleftrightarrow X(\omega - \omega_0)$$

(6) Differentiation in frequency domain

$$\text{If } x(n) \longleftrightarrow X(\omega) \\ \text{then } -jn x(n) \longleftrightarrow \frac{d}{d\omega} X(\omega)$$

(or)

$$n x(n) \longleftrightarrow j \frac{d}{d\omega} X(\omega)$$

(7) Convolution

$$\text{If } x_1(n) \longleftrightarrow X_1(\omega) \text{ \& } x_2(n) \longleftrightarrow X_2(\omega) \text{ then} \\ x_1(n) * x_2(n) \longleftrightarrow X_1(\omega) \cdot X_2(\omega)$$

* Energy spectral density (or) Rayleigh's Energy Theorem: It states that in a signal even if transformation is taking place the energy in the signal keeps constant

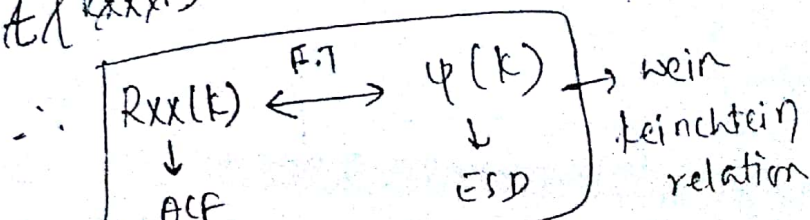
$$E_x(n) = \sum_{n=-\infty}^{\infty} |x(n)|^2 \neq \text{---}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

$$= \int_{-\infty}^{\infty} |x(f)|^2 df$$

$$\text{where } |X(\omega)|^2 = \psi(k) \\ \downarrow \\ \text{E.S.D}$$

$$\checkmark \quad E_x(k)$$



Ex. D $H(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(k) dk$

Energy of signal $x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(k) dk$

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1. for the given system response $y(n) = \frac{1}{2}x(n) + x(n-1) + \frac{1}{2}x(n-2)$
find its frequency response and also sketch its
mag & phase response.

Sol:- $y(n) = \frac{1}{2}x(n) + x(n-1) + \frac{1}{2}x(n-2)$

DFT $X(\omega) = \frac{1}{2}X(\omega) + e^{-j\omega}X(\omega) + \frac{1}{2}e^{-2j\omega}X(\omega)$

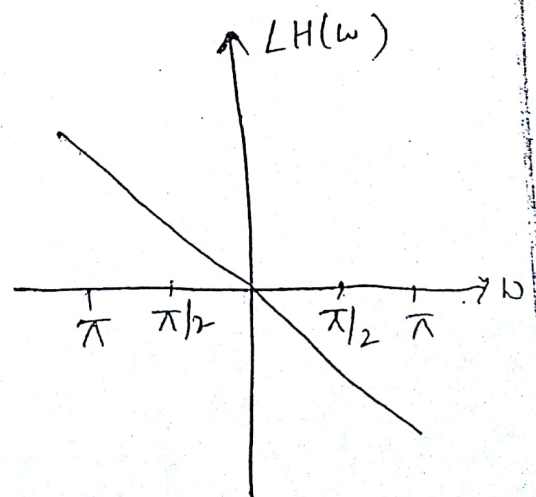
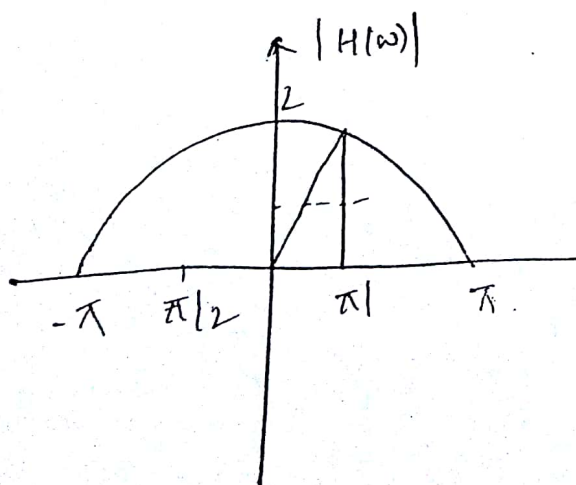
$$H(\omega) = \frac{1}{2} + e^{-j\omega} + \frac{e^{-2j\omega}}{2}$$

$$= e^{-j\omega} \left[1 + \frac{e^{j\omega} + e^{-j\omega}}{2} \right] = e^{-j\omega} (1 + \cos\omega)$$

$$|H(\omega)| = \sqrt{H(\omega) H^*(\omega)}$$

$$= |e^{-j\omega}| |1 + \cos\omega| = 1 + \cos\omega$$

$$\angle H(\omega) = 0 + \angle -\omega = -\omega$$



2. Find the F.T of the signal $x(n) = na^n u(n)$ $0 < a < 1$

Sol:- $X(j\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = \sum_{n=0}^{\infty} n(ae^{-j\omega})^n u(n)$

$$= \sum_{n=0}^{\infty} n(ae^{-j\omega})^n$$

$$j \cdot \frac{d}{d\omega} X(\omega) = j \cdot \frac{d}{d\omega} \frac{1}{(1 - ae^{-j\omega})} = \frac{-j(-ae^{-j\omega}(-j))}{1 - ae^{-j\omega}}$$

$$= \frac{ae^{-j\omega}}{1 - ae^{-j\omega}}$$

$$X(j\omega) = \frac{ae^{-j\omega}}{1 - ae^{-j\omega}} = \frac{a}{e^{j\omega} - a}$$

$$|X(\omega)| = \sqrt{\frac{ae^{-j\omega}}{1 - ae^{-j\omega}} \cdot \frac{ae^{j\omega}}{1 - ae^{j\omega}}}$$

$$= \frac{a}{\sqrt{a^2 - 2a\cos\omega + 1}}$$

$$\angle X(\omega) = \alpha(-\omega) - \tan^{-1}\left(-\frac{a}{\omega}\right)$$

$$= -\omega a + \tan^{-1}\left(\frac{a}{\omega}\right)$$

3. If $x(n) = \left(\frac{1}{2}\right)^n u(n)$ & $y(n) = x^2(n)$

Sol:- $x(n) = \left(\frac{1}{2}\right)^n u(n)$

$$X(\omega) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-j\omega n} = \frac{1}{1 - \frac{1}{2}e^{-j\omega}} = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

$$= \frac{2}{2 - e^{-j\omega}}$$

$$g(n) = x(n)$$

$$y(n) = \left(\frac{1}{4}\right)^n u(n)$$

$$Y(\omega) = \frac{1}{1 - \frac{1}{4}e^{-j\omega}} = \frac{4}{4 - e^{-j\omega}}$$

$$Y(e^{j\omega})^* = Y(\omega)^* = 4/3.$$

4. The frequency response of an ideal band pass filter is defined as $H(\omega) = 1$; $\pi/2 < |\omega| < 3\pi/2$
 $= 0$ else where then the energy of the

corresponding impulse response of the filter is

Solr

$$H(\omega) = 1, \quad \frac{\pi}{2} < |\omega| < 3\pi/2$$

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |x(\omega)|^2 d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega$$

$$= \frac{1}{2\pi} \int_{-3\pi/2}^{-\pi/2} 1 d\omega + \int_{\pi/2}^{3\pi/2} 1 d\omega$$

$$= \frac{1}{2\pi} (\pi + \pi) = 1J$$

Linear constant Difference equation:

$$\sum_{k=0}^M a_k y(n-k) = \sum_{k=0}^r b_k x(n-k)$$

$$a_0 y(n) = \sum_{k=1}^M a_k y(n-k) = \sum_{k=0}^r b_k z(n-k)$$

$$y(n) = \sum_{k=0}^r \frac{b_k z(n-k)}{a_0} = \sum_{k=1}^M \frac{a_k y(n-k)}{a_0}$$

1. If $x(n) = x_1(n)$ then $y(n) = a^k y(n)$ the system is said to LSI (linear shift invariant) or LTI (linear time invariant)

2. If $x(n) = 0 \quad n < 0$ & $y(n) = 0 \quad n < 0$ then the system is absolutely causal if ① & ② are satisfied then the system is absolutely stable.

NOTE: Inspite of having a perfectly stable response using DTFT but the digital signal processors cannot analyse a continuous frequency response, so the continuous frequency response in DTFT is converted into a discrete frequency response known as DFT.

*

DPS
(Discrete fourier series)

The F.T of periodic signal is taken as

$$\begin{aligned} \text{DPS} [x(n)] &= \tilde{x}(k) \\ \tilde{x}(k) &= \sum_{n=0}^{N-1} \tilde{x}(n) e^{-j(\frac{2\pi}{N})nk} \\ &= \sum_{n=0}^{N-1} \tilde{x}(n) \omega_N^{nk} \end{aligned}$$

DFT

(Discrete fourier transform)

The F.T of non-periodic signal is

$$\begin{aligned} \text{DFT} [x(n)] &= X(k) \\ X(k) &= \sum_{n=0}^{N-1} x(n) e^{-j(\frac{2\pi}{N})nk} \\ &= \sum_{n=0}^{N-1} x(n) \omega_N^{nk} \end{aligned}$$

ω_N^{nk} = twiddle factor representation

$$\omega_N^k = e^{j(\frac{2\pi}{N})k}$$

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Inverse (DFS)

1. DFS $[\tilde{x}(k) = \tilde{x}(n)]$

$$\tilde{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{x}(k) e^{j(\frac{2\pi}{N})nk}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \tilde{x}(k) \omega_N^{-nk}$$

$$\omega_N^k = e^{j(\frac{2\pi}{N})k}$$

↳ twiddle factor

Note: The frequency response of DFT or DFS is always a discrete frequency response.

Q. Calculate the DFT of the signal $x(n)=1$ where $0 \leq n \leq 2$ & also sketch its magnitude & phase response.

Sol: $X(k) = \sum_{n=0}^{N-1} x(n) e^{-j(\frac{2\pi}{N})nk}$

$$x(n)=1; 0 \leq n \leq 2$$

$$= \{1, 1, 1, 0\} \quad (\because N=4)$$

$$X(k) = \sum_{n=0}^3 x(n) e^{-j(\frac{2\pi}{4})nk}$$

$$= x(0) + x(1) e^{-j(\frac{\pi}{2})k} + x(2) e^{-j\pi k} + x(3) e^{-j(\frac{3\pi}{2})k}$$

$$= 1 + e^{-j\frac{\pi}{2}k} + e^{-j\pi k}$$

$$0 \leq n \leq N-1$$

$$0 \leq k \leq N-1 \quad 0 \leq k \leq 3$$

$$X(0) = 1+1+1=3$$

$$X(1) = 1+(-j)+(-1)=-j \quad X(2) = 1+(-1)+1=j$$

$$X(3) = 1+1+1=3$$

$$X(k) = \{3, -j, j, 3\}$$

$$X(|H|) = \{3\angle 0^\circ, j\angle -\pi/2, j\angle 0, 3\angle \pi/2\}$$

Inverse (DFT)

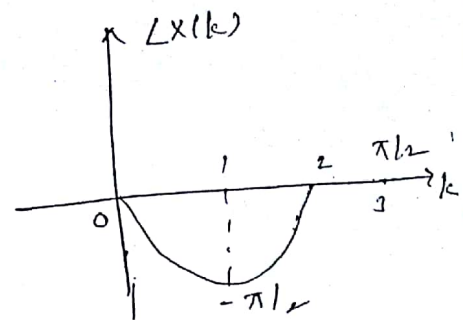
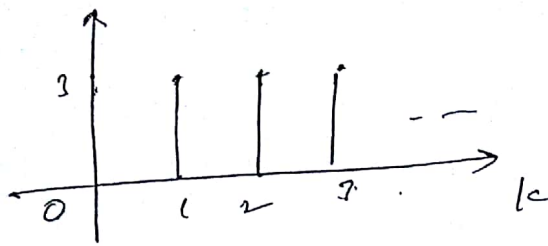
$$\text{IDFT} [x(k)] = x(n)$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} x(k) e^{j(\frac{2\pi}{N})nk}$$

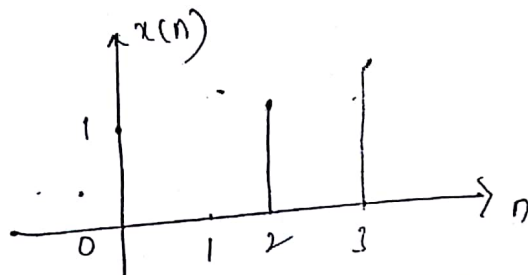
N = no. of discrete points in a sample

Q.

$$\angle x(k) = \left\{ 0, -\frac{\pi}{2}, 0, \frac{\pi}{2} \right\}$$



Q. calculate 4 pt DFS of i/p response $x(n) = \{1, 0, 2, 3\}$



$$\tilde{X}(k) = \sum_{n=0}^3 \tilde{x}(n) e^{-j\left(\frac{2\pi}{4}\right)nk}$$

$$= \tilde{x}(0) e^{-j0} + \tilde{x}(1) e^{-j\left(\frac{\pi}{2}\right)k} + \tilde{x}(2) e^{-j\pi k} + \tilde{x}(3) e^{-j\left(\frac{3\pi}{2}\right)k}$$

$$= 1 + 2e^{-j\pi k} + 3e^{-j\frac{3\pi}{2}k}$$

$$\tilde{X}(0) = 1 + 2 + 3 = 6$$

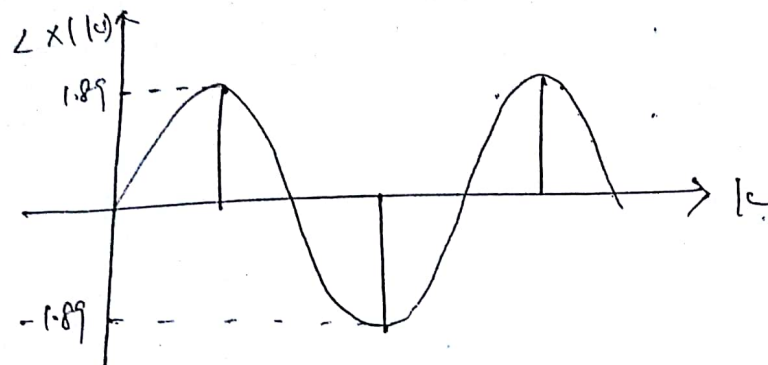
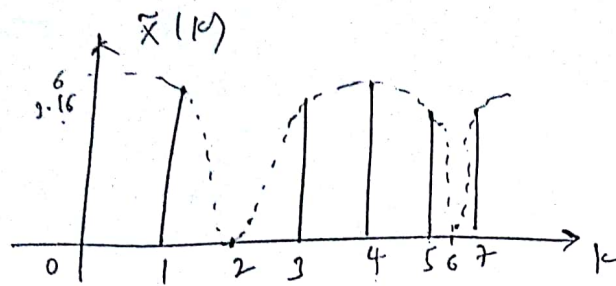
$$\tilde{X}(1) = 1 - 2 + 3j = 3j - 1$$

$$\tilde{X}(2) = 1 + 2 - 3 = 0$$

$$\tilde{X}(3) = 1 - 2 - 3j = -1 - 3j$$

$$\tilde{x}(k) = \{ \dots, -1-j, 6, 3j-1, 0, -1-3j, 6, 3j-1, 0, \dots \}$$

$$|\tilde{x}(k)| = \{ \dots, 6\angle 0, 3.16\angle 1.89, 0, 3.16\angle -1.89, \dots \}$$



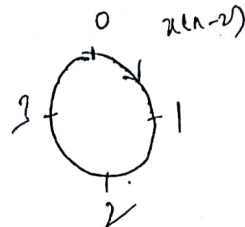
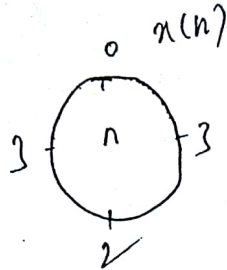
<u>S.No</u>	<u>Name of the property</u>	<u>DFS</u>	<u>DTF</u>
1.	Linearity property	$a\tilde{x}_1(n) + b\tilde{x}_2(n) \leftrightarrow a\tilde{x}_1(k) + b\tilde{x}_2(k)$	$aX_1(n) + bX_2(n) \leftrightarrow aX_1(k) + bX_2(k)$
2.	Periodicity	$\tilde{x}(n+N) \leftrightarrow \tilde{x}(n)$	
3.	Conjugate	$\tilde{x}^*(k) = \tilde{x}^*(N-k)$ (Circular shifting)	$X(k) = X^*(N-k)$ (Linear shifting)
4.	Shifting	$\tilde{x}(n+m) \leftrightarrow e^{j(\frac{2\pi}{N})mk} \tilde{x}(k, \text{Mod } N)$ $\leftrightarrow e^{-jm} \tilde{x}(k)$	$x(n+m) \leftrightarrow e^{j(\frac{2\pi}{N})mk} X(k)$ $\leftrightarrow W_N^m X(k)$
5.	Shifting in freq.	$\left[e^{j(\frac{2\pi}{N})ln} \tilde{x}(n) \right] \leftrightarrow \tilde{x}(k-l, \text{Mod } N)$ $\left[W_N^{-l} \tilde{x}(n) \right] \leftrightarrow \tilde{x}(k-l, \text{Mod } N)$	$e^{j(\frac{2\pi}{N})ln} x(n) \leftrightarrow X(k-l)$
6.	Convolution	<u>Circular convolution</u> $\tilde{x}_1(n) * \tilde{x}_2(n) \leftrightarrow \tilde{x}_1(k) \times \tilde{x}_2(k, \text{Mod } N)$	<u>Linear convolution</u> $[x_1(n) * x_2(n)] \leftrightarrow X_1(k) \times X_2(k)$

Proof ②: $\tilde{x}(k+N) \leftrightarrow \sum_{n=0}^{N-1} \tilde{x}(n) e^{-j \frac{2\pi}{N} n(k+N)}$

$$\leftrightarrow \sum_{n=0}^{N-1} \tilde{x}(n) e^{-j \frac{2\pi}{N} nk} \cdot \underbrace{e^{-j \frac{2\pi}{N} nN}}_{\rightarrow 1}$$

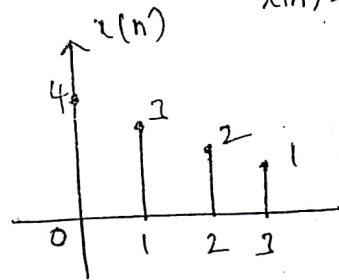
$$\leftrightarrow x(k)$$

Circular shifting:



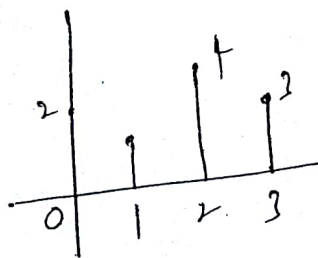
For the given signal $x(n)$ find the linear shifting $x(n-2)$ and circular shifting $x(n-2)$.

$$x(n) = \{4, 3, 2, 1\}$$

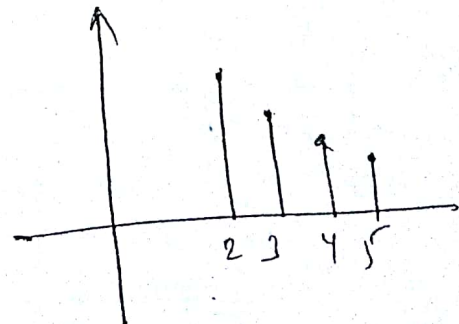
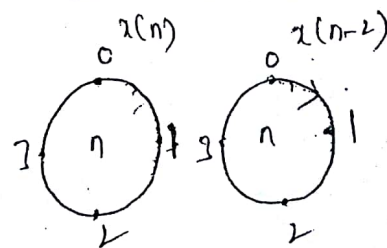


circular shifting

$$[x(n-2), \text{Mod } N]$$



circular shifting



Linear shifting

$$x(n+m) \leftrightarrow e^{j \frac{2\pi}{N} mk} x(k)$$

$$\leftrightarrow W_N^{mk} x(k)$$

Problems

* (k) The

find

Sol 1-

(2) 8

Problems

* (1) The DFT of a signal $x(n)$ is given as $X(k) = \{3, 2+3j, -5, x(3)\}$
find the values of $x(3)$ and $x(0)$

Sol: $X(k) = \{3, 2+3j, -5, x(3)\}$

$$N = 4$$

$$X(k) = X^*(N-k)$$

$$x(3) = X^*(4-3) \\ = X^*(1) = 2-3j$$

$$x(n) = \frac{1}{N} \left[\sum_{k=0}^{N-1} x(k) e^{-j\left(\frac{2\pi}{N}\right)nk} \right] \\ = \frac{1}{N} [x(0) + x(1) + x(2) + x(3)] \\ = \frac{1}{N} [3 + 2+3j - 5 + 2-3j] \\ = \frac{1}{4} [2] = \frac{1}{2} = 0.5$$

(2) 8 point DFT of a signal is given as $x(k)$ where
 $x(0)=3, x(1)=2+3j, x(2)=2-3j, x(3)=5, x(4)=3-5j$. Find
the values of $x(5), x(6), x(7)$

Sol: $x(5) = X^*(8-5) \\ = X^*(3) = 5$

$$x(6) = X^*(8-6) \\ = X^*(2) = 2+3j$$

$$x(7) = X^*(8-7) \\ = X^*(1) = 2-3j$$

(1) Graphical method for circular convolution:

→ Find the o/p response of the sequences $x(n) = \{2, 1, 2, 1, 2, 1, \dots\}$
 $\xleftarrow{N=4}$
 convolved with $h(n) = \{1, 2, 3, 4, 1, 2, 3, 4, \dots\}$
 $\xleftarrow{N=4}$

Concentric circles method:

$N \leftrightarrow$ maximum length of the sequence

Limit of o/p response

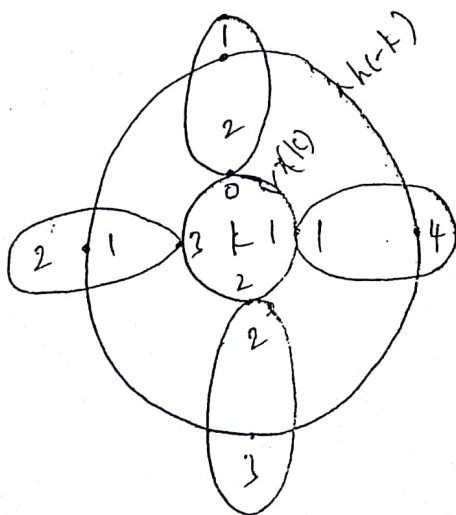
$$0 \leq n \leq N-1$$

$$0 \leq n \leq 3$$

$$y(n) = x(n) \otimes h(n)$$

$$= \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

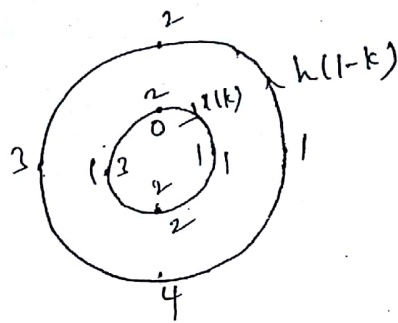
$$y(0) = \sum x(k) h(-k)$$



$$y(0) = 2 \times 2 + 1 \times 4 + 2 \times 2 + 2 \times 1$$

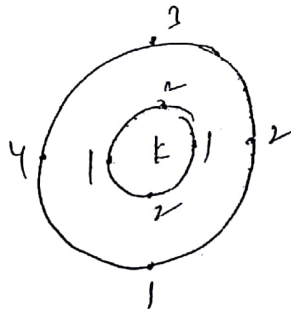
$$= 14$$

$$y(1) = \sum x(k) h(1-k)$$



$$y(1) = 4 + 1 + 8 + 3 = 16$$

$$y(2) = \sum x(k) h(2-k)$$



$$= 6 + 2 + 2 + 4 = 14$$

$$y(3) = 16$$

$$y(n) = \{14, 16, 14, 16, \dots\}$$

Tabular Method

k	-3	-2	-1	0	1	2	3
x(k)				2	4	2	1
h(-k)	4	3	2	1	4	3	2
h(1-k)	X	4	3	2	1	4	3
h(2-k)	X	X	4	3	2	1	4
h(3-k)	X	X	X	4	3	2	1

$$y(0) = \sum x(k) h(-k) = 2 \times 1 + 1 \times 4 + 2 \times 3 + 1 \times 2$$

$$y(1) = \sum x(k) h(1-k) = 14$$

$$y(2) = 14$$

$$y(3) = 16$$

Matrix Method

$$y(n) = x(n) \otimes h(n)$$

$$= [2 \ 1 \ 2 \ 1]$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{bmatrix}$$

$$= [14, 16, 14, 16]$$

18/3/15

* Find the system response when the signal $x(n) = \{1, 2\}$ convolved with $h(n) = \{3, 5\}$ using DFT method.

Soln

$$x(n) = \{1, 2\}$$

$$h(n) = \{3, 5\}$$

$$y(n) = x(n) * h(n)$$

$$Y(k) = X(k) \times H(k)$$

o/p limits are given by limit rule

$$0 + 0 \leq n \leq 1 + 1$$

$$0 \leq n \leq 2$$

$$N = 3$$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j \left(\frac{2\pi}{N} \right) nk}$$

$$X(k) = x(0) + x(1) e^{-j \left(\frac{2\pi}{3} \right) k} = \sum_{n=0}^2 x(n) e^{-j \left(\frac{2\pi}{3} \right) nk}$$

$$x(0) = 1 + 2 = 3$$

$$x(1) = 1 + 2e^{-j \left(\frac{2\pi}{3} \right)}$$

$$x(2) = 1 + 2e^{-j \left(\frac{4\pi}{3} \right)}$$

$$= \sum_{n=0}^2 x(n) e^{-j \left(\frac{2\pi}{3} \right) nk}$$

$$x(0) = x(0) + x(1) e^{-j \left(\frac{2\pi}{3} \right) 0 \times 1}$$

$$= x(0) + x(1) = 3$$

$$x(1) = x(0) + x(1) e^{-j \frac{2\pi}{3} \times 1 \times 1}$$

$$= x(0) + x(1) e^{-j \frac{2\pi}{3}}$$

$$x(2) = x(0) + x(2) e^{-j \frac{2\pi}{3} \times 1 \times 2}$$

$$= x(0) + x(2) e^{-j \frac{4\pi}{3}}$$

cont. prob 4

$$x(1) = x(0) + x(1) \left[\cos \frac{2\pi}{2} + j \sin \frac{2\pi}{2} \right]$$

$$= x(0) + x(1) \left[\frac{-1}{2} - \frac{j\sqrt{3}}{2} \right] = -1.722j$$

$$x(1) = x(0) + x(2) \left[\cos \frac{4\pi}{2} - j \sin \frac{4\pi}{2} \right]$$

$$= 1 + 2 \left[\frac{-1}{2} + \frac{j\sqrt{3}}{2} \right] = 1.722j$$

$$H(0) = h(0) + h(1) = 8$$

$$H(1) = h(0) + h(1) e^{-j\frac{2\pi}{2}}$$

$$= 2 + 5 \left(e^{-j\frac{2\pi}{2}} \right) = 0.5 - 4.5j$$

$$H(2) = h(0) + h(2) e^{-j\frac{4\pi}{2}}$$

$$= 2 + 5 \left[e^{-j\frac{4\pi}{2}} \right] = 0.5 + 4.5j$$

$$y(0) = x(0) \cdot H(0) = 24$$

$$y(1) = x(1) \cdot H(1) = -7.49 - 0.86j$$

$$y(2) = x(2) \cdot H(2) = -7.49 + 0.86j$$

$$y(k) = \{ 24, -7.49 - 0.86j, -7.49 + 0.86j \}$$

$$y(n) = \text{inverse DFT} \{ y(k) \}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} x(k) e^{j \left(\frac{2\pi}{N} \right) nk}$$

$$y(0) = \frac{1}{3} \sum_{k=0}^2 x(k) e^{j \frac{2\pi}{3} k \cdot 0}$$

$$= x(0) + x(1) + x(2)$$

$$= 24 - 7.49 - 0.86j + 7.49 + 0.86j = 24$$

}

$$\begin{array}{c|ccc} & 1 & 2 & \\ \hline 3 & 2 & 8 & \\ 5 & 5 & 16 & \end{array}$$

Find the circular convolution of the inputs $x(n) = \{3, -1, 2\}$ convolved with $h(n) = \{1, 1, -1\}$ using DFT method.

Sol:-

$$x(n) = \{3, -1, 2\}$$

$$h(n) = \{1, 1, -1\}$$

Limits of o/p circular convolution is

$$0 \leq n \leq N-1 \Rightarrow N=3$$

$$0 \leq n \leq 2$$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\left(\frac{2\pi}{N}\right)nk}$$

$$= \sum_{n=0}^2 x(n) e^{-j\left(\frac{2\pi}{3}\right)nk}$$

$$x(k) = x(0) + x(1) e^{-j\left(\frac{2\pi}{3}\right)k} + x(2) e^{-j\left(\frac{4\pi}{3}\right)k}$$

$$= 3 - 1 e^{-j\left(\frac{2\pi}{3}\right)k} + 2 e^{-j\left(\frac{4\pi}{3}\right)k}$$

$$x(0) = 3 - 1 + 2 = 4$$

$$x(1) = 3 - 1 e^{-j\left(\frac{2\pi}{3}\right)} + 2 e^{-j\left(\frac{4\pi}{3}\right)}$$

$$= 3 - 1 \left(\cos \frac{2\pi}{3} - j \sin \frac{2\pi}{3} \right) + 2 \left(\cos \frac{4\pi}{3} - j \sin \frac{4\pi}{3} \right)$$

$$= 3 - 1 \left(-\frac{1}{2} - j \frac{\sqrt{3}}{2} \right) + 2 \left(-\frac{1}{2} + j \frac{\sqrt{3}}{2} \right)$$

$$= 2.5 + 2.598j$$

$$x(z) = 3 - 1e^{-j(\frac{4\pi}{3})} + 2e^{-j(\frac{8\pi}{3})}$$

Z-Transforms

⇒ The major drawback with DTFT is only stable systems can be analysed so, in order to analyse both stable and unstable systems, bounded and unbounded signals Z-Transforms are used rather than DTFT.

$$Z.T [x(n)] = X(z)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \rightarrow Z-T$$

Let z be a complex exponential

$$z = re^{j\omega}$$

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) (re^{j\omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\omega n}$$

If $r=1$ then

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \rightarrow DTFT$$

⇒ $z = e^{j\omega} \rightarrow$ relation between Z-T & DTFT.

*NOTE: 1. Any continuous time signal can be expressed as complex frequencies. $[s = j\omega]$

*2. Any discrete time signal can be expressed for complex exponential frequencies $[z = e^{j\omega}]$

* Inverse Z-Transforms:

$$\text{Inv. Z.T } [X(z)] = x(n)$$

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

- Cauchy's integral
- " Residue
- Power series expansion
- Partial fractions

* Find the Z-Transforms for the given input signal and also indicate its region of convergence

(1) $x(n) = a^n u(n)$

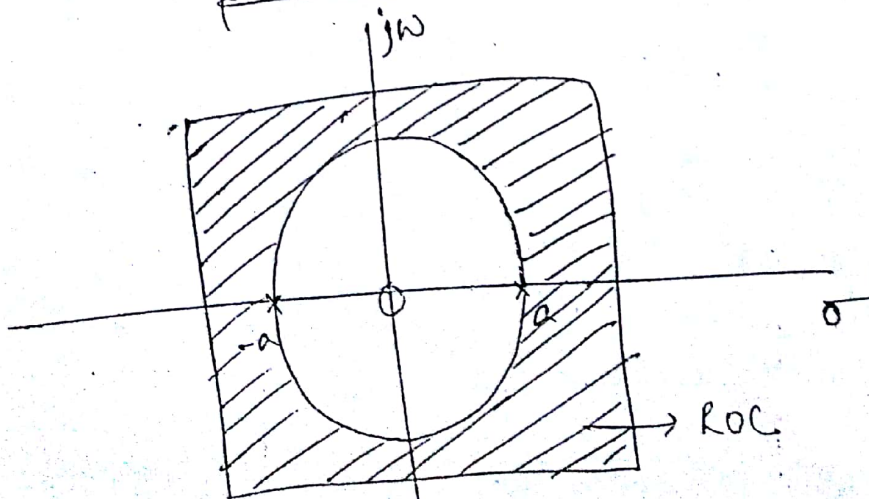
$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

$$X(z) = \frac{1}{1 - az^{-1}} = \left[\frac{z}{z-a} \right]$$

$$az^{-1} < 1$$

$$|z| > a \rightarrow \text{ROC}$$



(2)

$$x(n) = -a^n u(-n-1)$$

$$X(z) = - \sum_{n=-\infty}^{\infty} a^n u(-n-1) z^{-n}$$

$$= - \sum_{n=-\infty}^{-1} (az^{-1})^n$$

$$= - \sum_{n=1}^{\infty} (az^{-1})^{-n}$$

$$= - \sum_{n=1}^{\infty} (a^{-1}z)^n$$

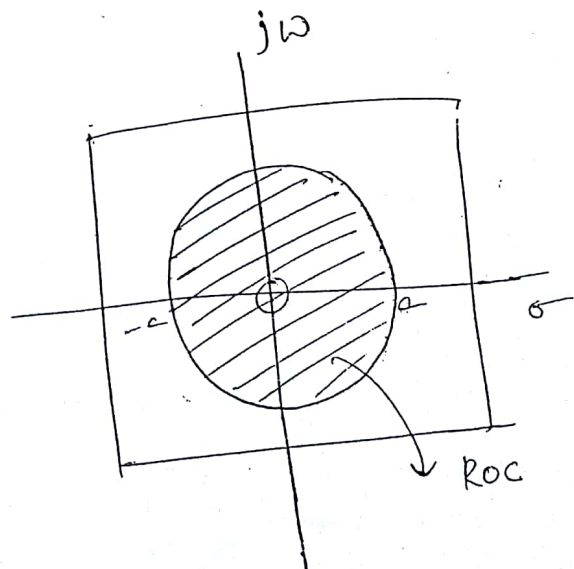
$$= - \left[\sum_{n=0}^{\infty} (a^{-1}z)^n - 1 \right]$$

$$= 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$

$$= 1 - \frac{1}{(1 - a^{-1}z)} = \frac{z}{z-a}$$

$$a^{-1}z < 1$$

$$|z| < a$$



* Regi
the
Co:

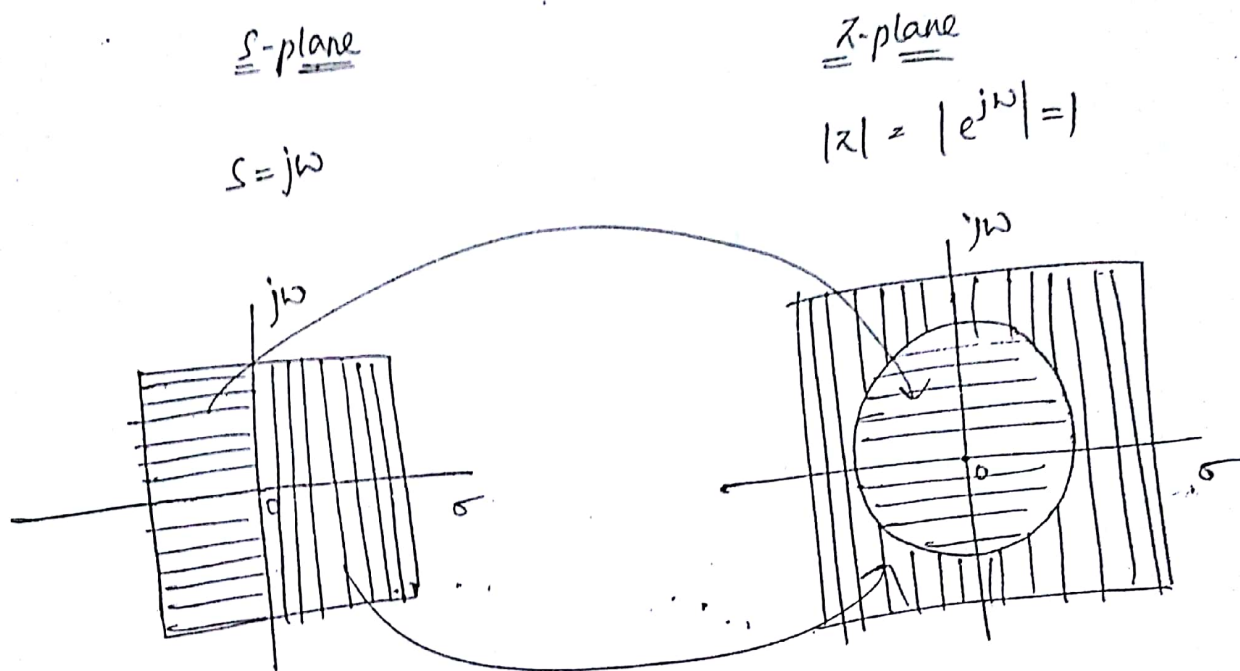
* Relati
Inte

* find
the

(1)

* Region of Convergence: The range of values of 'z' for which the signal $x(n)$ converges is known as the region of convergence.

* Relation between s-plane and z-plane and mapping of s-plane into z-plane



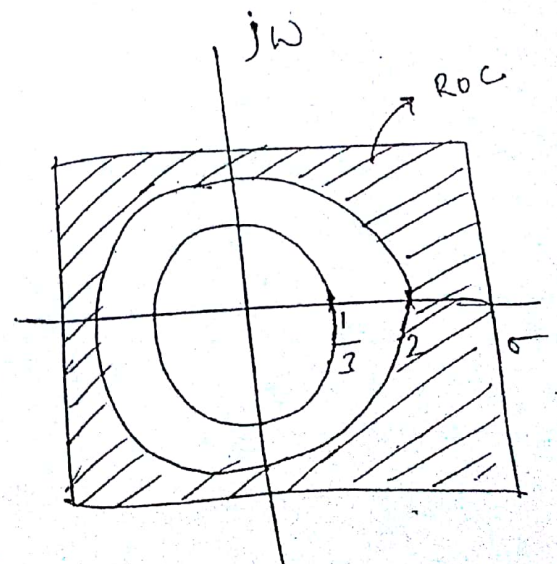
* Find the z-transform for the following sequences and indicate their ROC

(1) $x(n) = 2^n u(n) + 3^{-n} u(n)$
 $= 2^n u(n) + \left(\frac{1}{3}\right)^n u(n)$

$$X(z) = \frac{1}{(1-2z^{-1})} + \frac{1}{(1-\frac{1}{3}z^{-1})}$$

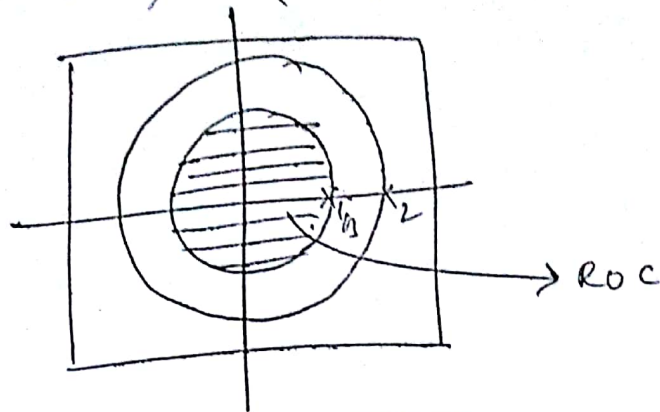
$$= \left[\frac{z}{z-2} \right] + \left[\frac{z}{z-\frac{1}{3}} \right]$$

$|z| > 2$



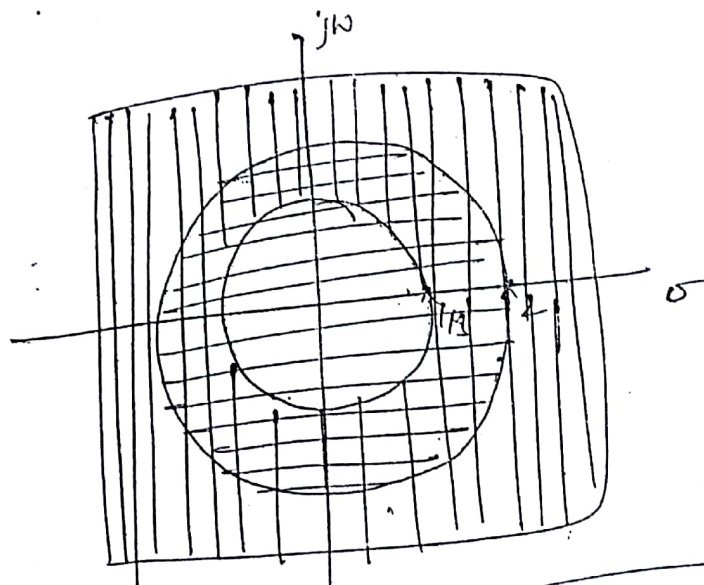
2. $x(n) = -2^n u(-n-1) - 3^{-n} u(-n-1)$

$$X(z) = \left(\frac{z}{z-2} \right) + \left(\frac{z}{z-\frac{1}{3}} \right)$$



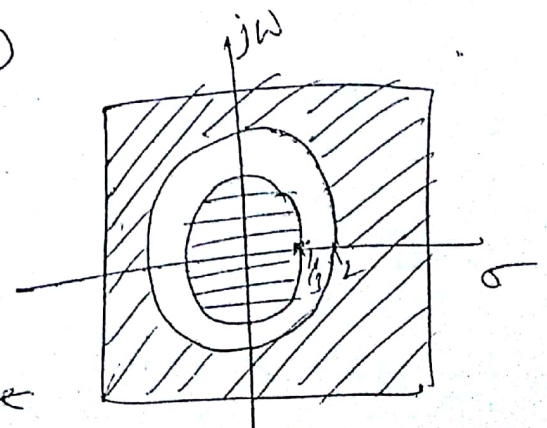
$$\boxed{|z| < \frac{1}{3}} \rightarrow \text{Roc}$$

3. $x(n) = -2^n u(-n-1) + 3^{-n} u(n)$



$$\boxed{\frac{1}{3} < |z| < 2} \rightarrow \text{Roc}$$

4. $x(n) = 2^n u(n) - 3^{-n} u(-n-1)$



As there is no common region

$\therefore$ Roc does not exist

* Find the z-transform of the following signal and also

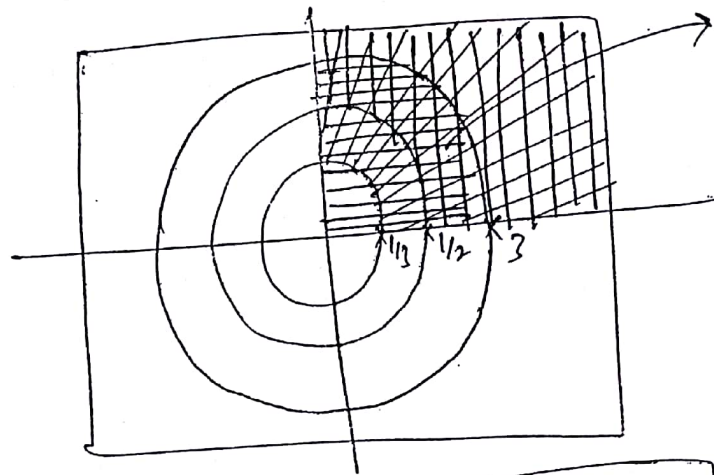
Indicate its ROC.

(1) $x(n) = \left(\frac{1}{3}\right)^{|n|} + \left(\frac{1}{2}\right)^n u(n)$

Sol:-

$$= \left(\frac{1}{3}\right)^n u(n) + \left(\frac{1}{3}\right)^{-n} u(-n-1) + \left(\frac{1}{2}\right)^n u(n)$$

$$= \left(\frac{1}{3}\right)^n u(n) + 3^n u(-n-1) + \left(\frac{1}{2}\right)^n u(n)$$



common ROC
b/w all the
poles.

$$\boxed{\frac{1}{2} < |z| < 3}$$

$$X(z) = \left[\frac{z}{z - \frac{1}{3}} \right] - \left[\frac{z}{z - 3} \right] + \left[\frac{z}{z - \frac{1}{2}} \right]$$

* $x(n) = \delta(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n}$$

$$= 1$$

Roc is entire
z-plane

$$x(n) = u(n)$$

$$X(z) = \sum_{n=-\infty}^{\infty} u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} (z^{-1})^n$$

$$= \frac{1}{1 - z^{-1}}$$

$$= \frac{z}{z - 1}$$

$$\boxed{|z| > 1}$$

$$x(n) = -u(-n-1)$$

$$X(z) = \frac{z}{z - 1}$$

$$\boxed{|z| < 1}$$

$$x(n) = \cos \omega_0 n u(n)$$

$$X(z) = \sum_{n=0}^{\infty} \cos \omega_0 n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \cos \omega_0 n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left[\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2} \right] z^{-n}$$

$$= \frac{1}{2} \left[\sum_{n=0}^{\infty} e^{j\omega_0 n} z^{-n} + \sum_{n=0}^{\infty} e^{-j\omega_0 n} z^{-n} \right]$$

$$= \frac{1}{2} \left[\frac{1}{1 - \frac{e^{j\omega_0}}{z}} + \frac{1}{1 - \frac{e^{-j\omega_0}}{z}} \right]$$

$$= \frac{1}{2} \left[\frac{z}{z - e^{j\omega_0}} + \frac{z}{z - e^{-j\omega_0}} \right]$$

$$= \frac{z}{2} \left[\frac{1}{z - e^{j\omega_0}} + \frac{1}{z - e^{-j\omega_0}} \right]$$

$$= \frac{z}{2} \left[\frac{z - e^{-j\omega_0} + z - e^{j\omega_0}}{(z - e^{j\omega_0})(z - e^{-j\omega_0})} \right]$$

$$= \frac{z}{2} \left[\frac{2z - (e^{-j\omega_0} + e^{j\omega_0})}{z^2 - ze^{-j\omega_0} - ze^{j\omega_0} + 1} \right]$$

$$= \frac{z}{2} \left[\frac{2z - (2 \cos \omega_0)}{z^2 - 2z \cos \omega_0 + 1} \right]$$

$$= \frac{z(z - \cos \omega_0)}{z^2 - 2z \cos \omega_0 + 1}$$

$$\frac{z(z - \cos \omega_0)}{(z^2 - 2z \cos \omega_0 + 1)}$$

$$\frac{z \sin \omega_0}{(z^2 - 2z \cos \omega_0 + 1)}$$

$$|z| > 1$$

$$x(n) = \sin \omega_0 n u(n)$$

$$X(z) = \sum_{n=-\infty}^{\infty} \sin \omega_0 n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \sin \omega_0 n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left[\frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2} \right] z^{-n}$$

$$= \frac{1}{2} \left[\sum_{n=0}^{\infty} e^{j\omega_0 n} z^{-n} - \sum_{n=0}^{\infty} e^{-j\omega_0 n} z^{-n} \right]$$

$$= \frac{1}{2} \left[\frac{1}{1 - \frac{e^{j\omega_0}}{z}} - \frac{1}{1 - \frac{e^{-j\omega_0}}{z}} \right]$$

$$= \frac{1}{2} \left[\frac{z}{z - e^{j\omega_0}} - \frac{z}{z - e^{-j\omega_0}} \right]$$

$$= \frac{z}{2} \left[\frac{\cancel{z - e^{-j\omega_0}} \rightarrow z + e^{j\omega_0}}{z^2 - ze^{-j\omega_0} - ze^{j\omega_0} + 1} \right]$$

$$= \frac{z}{2} \left[\frac{e^{j\omega_0} - e^{-j\omega_0}}{z^2 - 2z \cos \omega_0 + 1} \right]$$

$$= \frac{z}{2} \left[\frac{\cancel{2} \sin \omega_0}{z^2 - 2z \cos \omega_0 + 1} \right]$$

$$= \frac{z \sin \omega_0}{z^2 - 2z \cos \omega_0 + 1}$$

$|z| > 1$

23/3/19

<u>S.No</u>	<u>Signal</u>	<u>Z-Transforms</u>	<u>ROC</u>
1.	unit impulse $\delta(n)$	1	entire z-plane
2.	unit step $u(n)$	$\frac{1}{(1-z^{-1})}$ or $\frac{z}{(z-1)}$	$ z > 1$
3.	$-u(-n-1)$	11	$ z < 1$
4.	$nu(n)$	$\frac{z}{(z-1)^2}$	$ z > 1$
5.	$-nu(-n-1)$	11	$ z < 1$
*6.	$a^n u(n)$	$\frac{1}{(1-az^{-1})}$ or $\left(\frac{z}{z-a}\right)$	$ z > a$
7.	$-a^n u(-n-1)$	11	$ z < a$
8.	$na^n u(n)$	$\frac{az}{(z-a)^2}$	$ z > a$
9.	$-na^n u(-n-1)$	11	$ z < a$
10.	$\cos \omega_0 n u(n)$	$\frac{z(z - \cos \omega_0)}{(z^2 - 2z \cos \omega_0 + 1)}$	$ z > 1$
11.	$\sin \omega_0 n u(n)$	$\frac{z \sin \omega_0}{(z^2 - 2z \cos \omega_0 + 1)}$	$ z > 1$
12.	a^n	$0 \leq n \leq N-1$	ROC is entire z-plane except $z=0$ (or) $ z > 0$

$$\sum_{n=0}^{N-1} \frac{1}{(az^{-1})^n}$$

$$\frac{1}{(1-az^{-1})} = \frac{z}{z-a}$$

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n}$$

$$= \sum_{n=0}^{N-1} (az^{-1})^n$$

$$= \frac{1 - (az^{-1})^{N-1+1}}{1 - az^{-1}}$$

$$= \left[\frac{1 - a^N z^{-N}}{1 - az^{-1}} \right]$$

$$0 \leq n \leq N-1$$

* Inverse Z-Transforms:

$$\text{Inv. Z.T. } [X(z)] = x(n)$$

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

- (1) Partial fraction method
- (2) Long division
- (3) Cauchy's residue method.

for the given Z-Transform $X(z) = \frac{z - \frac{5}{6}z^{-1}}{(1 - \frac{1}{3}z^{-1})(1 - \frac{1}{4}z^{-1})}$. Find the inverse Z-transform using partial fraction method for

(a) if $X(z)$ is unilateral Z.T

(b) if $\frac{1}{4} < |z| < \frac{1}{3}$

(c) if $|z| < \frac{1}{4}$

Soln

$$X(z) = \frac{A}{1 - \frac{1}{3}z^{-1}} + \frac{B}{1 - \frac{1}{4}z^{-1}}$$

To find A sub. $1 - \frac{1}{3}z^{-1} = 0$

$$-1 = ?$$

$$A = \frac{3 - \frac{1}{2} \times \frac{1}{2}}{\left(1 - \frac{1}{4} \times 1\right)} = \frac{\frac{5}{4}}{\frac{3}{4}} = \frac{5}{3}$$

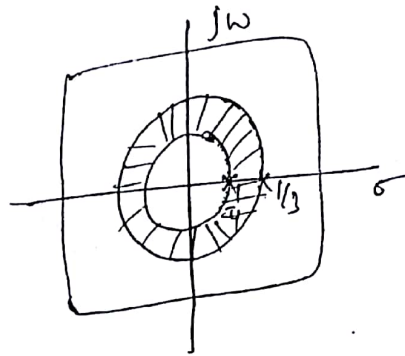
$$B = \frac{3 - \frac{5}{4} \times 1^2}{1 - \frac{1}{3} \times 1} = \frac{-\frac{1}{4}}{-\frac{2}{3}} = \frac{3}{4}$$

$$X(z) = \frac{2}{\left(1 - \frac{1}{3}z^{-1}\right)} + \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)}$$

(a) $X(z)$ is $1/2$ -T

$$x(n) = 2\left(\frac{1}{3}\right)^n u(n) + \left(\frac{1}{4}\right)^n u(n)$$

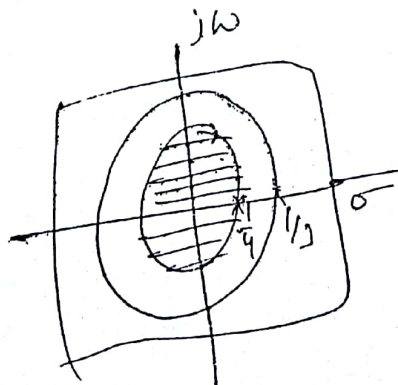
(b) $\frac{1}{4} < |z| < \frac{1}{3}$



$$x(n) = -2\left(\frac{1}{3}\right)^n u(-n-1) + \left(\frac{1}{4}\right)^n u(n)$$

(c) $|z| < \frac{1}{4}$

$$x(n) = -2\left(\frac{1}{3}\right)^n u(-n-1) - \left(\frac{1}{4}\right)^n u(-n-1)$$



* find the inverse z-transform of the signal $X(z) = \frac{z^2}{(z+2)(z-3)}$

Sol:-

$$X(z) = \frac{z^2}{(z+2)(z-3)}$$

$$\frac{X(z)}{z} = \frac{A}{(z+2)} + \frac{B}{(z-3)}$$

$$A = \frac{2}{5}, \quad B = \frac{3}{5}$$

$$\frac{X(z)}{z} = \frac{2}{5(z+2)} + \frac{3}{5(z-3)}$$

$$X(z) = \frac{2}{5} \left(\frac{z}{z+2} \right) + \frac{3}{5} \left(\frac{z}{z-3} \right)$$

$X(z)$ is v.z-T

$$x(n) = \frac{2}{5} (-2)^n u(n) + \frac{3}{5} (3^n) u(n)$$

* Inverse z-transforms by long divisions method:

(i) find the inverse z.T of the following signals using long division method or power series expansion

(a) $X(z) = \frac{z}{(2z^2 - 3z + 1)}$; $|z| > 1$

$$2z^2 - 3z + 1 \overline{) \left(\frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} \right)}$$

$$\underline{\frac{1}{2}z^{-1} + \frac{3}{4}z^{-2}}$$

$$\frac{7}{8}z^{-3} - \frac{1}{2}z^{-1}$$

$$\underline{\frac{7}{8}z^{-3} - \frac{9}{4}z^{-1} + \frac{3}{4}z^{-2}}$$

$$\frac{7}{4}z^{-1} - \frac{3}{4}z^{-2}$$

$$\underline{\frac{7}{4}z^{-1} - \dots}$$

$$x(n) = \left\{ \underset{\uparrow}{0}, \frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \dots \right\}$$

$$(b) \quad X(z) = \frac{z}{(2z^2 - 3z + 1)} \quad |z| < 1/2$$

$$\begin{array}{r} 1 - 3z + 2z^2 \overline{) 2z + 3z^2 + 7z^3} \\ \underline{2z - 3z^2 + 2z^3} \\ 3z^2 - 2z^3 \\ \underline{3z^2 - 9z^3 + 6z^4} \\ 7z^3 - 6z^4 \\ \underline{7z^3 - 21z^4 + 14z^5} \\ 15z^4 - 14z^5 \dots \end{array}$$

$$x(n) = \left\{ \dots, 7, 3, 1, \underset{\uparrow}{0} \right\}$$

$$(c) \quad X(z) = \frac{z^2 + z + 1}{(3z^2 + 2z + 2)}, \quad |z| > 0$$

$$\begin{array}{r} 3z^2 + 2z + 2 \overline{) z^2 + z + 1} \\ \underline{z^2 + \frac{2}{3}z + \frac{2}{3}} \\ \frac{z}{3} + \frac{1}{3} \\ \underline{\frac{1}{3}z + \frac{2}{9} + \frac{2}{92}} \\ \frac{1}{9} - \frac{2}{92} \\ \underline{\frac{1}{9} - \frac{22}{27} + \frac{2}{27}} \end{array}$$

$$x(n) = \left\{ \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \dots \right\}$$

* Inverse Z-transform by Cauchy's Residue Method

Inv. Z-T [X(z)]

$$x(n) = \sum_{\text{all the poles of } x(z)} \text{Residue} [X(z) \cdot z^{n-1}]$$

for the order m having a pole at $z = \beta$

$$\text{then the Residue} = \frac{1}{(m-1)!} \lim_{z \rightarrow \beta} \left[\frac{d^{m-1}}{dz^{m-1}} (z-\beta)^m x(z) z^{n-1} \right]$$

(1) find i. Z-T using residue method for the signal

$$X(z) = \frac{z}{(z-1)(z-2)} \quad (a) |z| > 2, \quad (b) |z| < 1$$

Sol:- $X(z) = \frac{z}{(z-1)(z-2)}$

for order $m=1$; pole at $z=1$

$$\text{Residue} = \frac{1}{(1-1)!} \lim_{z \rightarrow 1} \left[\frac{d^0}{dz^0} (z-1) \cdot \frac{z}{(z-1)(z-2)} z^{n-1} \right]$$

$$= -1$$

for order $m=1$, pole at $z=2$

$$\text{Residue} = \frac{1}{(1-1)!} \lim_{z \rightarrow 2} \left[\frac{d^0}{dz^0} (z-2) \cdot \frac{z}{(z-1)(z-2)} z^{n-1} \right]$$

$$= \lim_{z \rightarrow 2} \frac{z}{(z-1)} z^{n-1}$$

$$= \frac{2}{1} 2^{n-1} = 2^n$$

$$x(n) = \text{sum of all residues}$$

$$(a) \quad x(n) = (-1)^n u(n) + 2^n u(n)$$

$$(b) \quad x(n) = x(-n-1) - 2^n u(-n-1)$$

(2) find the Z.T of $x(z) = \frac{z}{(z+1)^2(z+2)^2}$, $|z| > 2$

Sol: for order $m=2$, pole at $z=-1$

$$\text{Residue} = \frac{1}{(2-1)!} \lim_{z \rightarrow -1} \left[\frac{d}{dz} \left(\cancel{(z+1)^2} \cdot \frac{z}{(z+1)^2(z+2)^2} z^{n-1} \right) \right]$$

$$= \lim_{z \rightarrow -1} \left[\frac{d}{dz} \frac{z}{(z+2)^2} \cdot \frac{z^n}{z} \right]$$

$$= \lim_{z \rightarrow -1} \left[\frac{(z+2)^2 n z^{n-1} - z^n 2(z+2)}{(z+2)^6} \right]$$

$$= \frac{(-1+2)^2 n z^{n-1} - (-1)^n 2(-1+2)^2}{(1+2)^6}$$

$$= \frac{-n(-1)^{n-1} - 2(-1)^n}{(1+2)^6} \Rightarrow \frac{(-1)^n (n+3)}{(1+2)^6}$$

for order $m=2$, pole at $z=-2$

$$\text{Residue} = \frac{1}{(2-1)!} \lim_{z \rightarrow -2} \left[\frac{d}{dz} \left(\cancel{(z+2)^2} \cdot \frac{z}{(z+1)^2(z+2)^2} z^n \right) \right]$$

$$= \frac{1}{2} \lim_{z \rightarrow -2} \left[\frac{d}{dz} \frac{z^n}{(z+1)^2} \right]$$

$$= \frac{1}{2} \left[\frac{d}{dz} \frac{(z+1)^2 n z^{n-1} - z^n 2(z+1)}{(z+1)^4} \right]$$

$$= \frac{1}{2} \left[\cancel{(z+1)^2} \cdot \frac{z^{n+1}}{z} \right]$$

$$= \frac{1}{2} \left[\frac{(z+1)^4 \left[n 2(z+1) (n-1) z^{n-2} - 2n z^{n-1} \right] - \left[(z+1)^4 n z^{n-1} - 2z^n (z+1) \right]}{(z+1)^4} \right]$$

$$\text{Residue} = -2^n \left[0.125n^2 - 1.125n + 1 \right]$$

$$x(n) = -(n+1) (-1)^n u(n) + \left[(0.125n^2 - 1.125n + 1) (-2)^n \right] u(n)$$

Properties of _____

1. Linearity:

$$\text{If } x_1(n) \longleftrightarrow X_1(z) \text{ and } x_2(n) \longleftrightarrow X_2(z) \text{ then}$$

$$ax_1(n) + bx_2(n) \longleftrightarrow aX_1(z) + bX_2(z).$$

2. Time Reversal:

$$\text{If } x(n) \longleftrightarrow X(z) \text{ then } x(-n) \longleftrightarrow X(z^{-1})$$

$$\rightarrow X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(-n) [z^{-(-n)}]$$

$$\text{Let } -n = k$$

$$= \sum_{k=-\infty}^{\infty} x(k) z^{-k}$$

$$= \underline{\underline{X(z^{-1})}}$$

* Find the z-T of the signal $x(n) = u(-n)$ using Time reversal property

$$x(n) = u(-n)$$

Sol:-

$$u(n) \longleftrightarrow \frac{1}{(1-z^{-1})}, |z| > 1$$

* fir

Sol 1-

$$x(-n) \leftrightarrow \frac{1}{(1 - (z^{-1})^{-1})} ; |z^{-1}| > 1$$

$$\leftrightarrow \frac{1}{(1 - z)} ; \underline{|z| < 1}$$

2. Multiplication of the sequence with an exponential

$$(1) \text{ If } x(n) \leftrightarrow X(z)$$

$$\text{then } a^n x(n) \leftrightarrow X(a^{-1}z) = X\left(\frac{z}{a}\right)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^n x(n) [z^{-n}]$$

$$= \sum_{n=-\infty}^{\infty} x(n) [a z^{-1}]^n$$

$$= X(a^{-1}z)$$

$$(2) \text{ If } x(n) \leftrightarrow X(z) \text{ then}$$

$$e^{j\omega_0 n} x(n) \leftrightarrow X\left[(e^{j\omega_0})^{-1} z\right] = X\left(\frac{z}{e^{j\omega_0}}\right)$$

- find the z-Transform of signal $x(n) = \left(\frac{1}{2}\right)^n \cos \omega_0 n u(n)$

$$X(z) = \left(\frac{1}{2}\right)^n \cos \omega_0 n u(n)$$

$$\cos \omega_0 n u(n) \leftrightarrow \frac{z(z - \cos \omega_0)}{(z^2 - 2z \cos \omega_0 + 1)}$$

$$a = \frac{1}{2}, \quad a^{-1} = 2$$

$$X(a^{-1}z) = X(2z)$$

$$= \frac{(2z) [2z - \cos \omega_0]}{[(2z)^2 - 2 \cdot 2z \cos \omega_0 + 1]}$$

$$= \frac{2z(2z - \cos \omega_0)}{(4z^2 - 4z \cos \omega_0 + 1)}$$

4. Time Shifting Property:

$$\text{If } x(n) \leftrightarrow X(z)$$

$$\text{then } x(n-n_0) \leftrightarrow z^{-n_0} X(z)$$

* Find the Z.T of the signal $y(n) = x_1(n+2) + x_2(-n+1)$ where $x_1(n) = \left(\frac{1}{2}\right)^n u(n)$ and $x_2(n) = \left(\frac{1}{3}\right)^n u(n)$, and also indicate its ROC.

Sol:

$$y(n) = x_1(n+2) + x_2(-n+1)$$

$$(x_1(z) \neq \left(\frac{1}{2}\right)^n)$$

$$x_1(z) = z^2 X(z) = z^2 \cdot \frac{z}{z - \frac{1}{2}} = z^3 \cdot \frac{z^2}{z^2 - 1}$$

$$(x_2(z) \neq \#)$$

$$x_2(-n+1) = x_2(-(n-1))$$

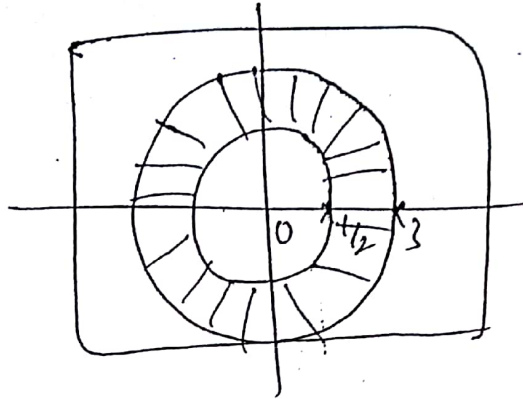
$$x_2(-n) \leftrightarrow x_2(z^{-1})$$

$$x_2[-(n-1)] \leftrightarrow z^{-1} x_2(z^{-1}) \rightarrow z^{-1} \cdot \left(\frac{1}{1 - \frac{1}{3}z} \right)$$

$$\left(\frac{1}{3}\right)^n u(n) \leftrightarrow \frac{1}{\left(1 - \frac{1}{3}z^{-1}\right)}$$

$$x_1(n) = \left(\frac{1}{2}\right)^n u(n) \leftrightarrow \frac{z}{z - \frac{1}{2}}$$

$$Y(z) = z^3 \times \frac{z}{\left(z - \frac{1}{2}\right)} - \frac{1}{z(z-3)}$$



$$\boxed{\frac{1}{2} < |z| < 3}$$

$$\downarrow$$

ROC

* Find Z-T of the signal ~~xxx~~ $x(n) = u(n-2) + u(5-n)$

Sol:-

$$x(n) = u(n-2) + u(5-n)$$

$$u(n-2) \leftrightarrow \begin{matrix} z^{-2} X(z) \\ z^{-2} \frac{z}{z-1} \end{matrix}$$

$$u(n) \leftrightarrow \frac{z}{z-1}$$

$$u(-n) \leftrightarrow \frac{1}{1-z}$$

$$X(z) = z^{-2} \times \frac{z}{z-1} - \frac{z^{-5}}{z-1}$$

$$= \frac{z^{-1} - z^{-5}}{z-1}$$

* Find the IZT of $X(z) = \frac{z^{-9}}{[z-9]}$ if $X(z)$ is U.Z.T.

Sol:-

$$X(z) = \frac{z^{-9}}{[z-9]}$$

$$= z^{-10} \left[\frac{z}{z-9} \right]$$

$$\frac{z}{(z-a)} \leftrightarrow a^n u(n)$$

$$z^{-10} \times \frac{z}{(z-a)} \leftrightarrow \underline{\underline{a^{n-10} u(n-10)}}$$

**5. Differentiation in z-domain:

$$x(n) \leftrightarrow X(z)$$

$$\text{then } n x(n) \leftrightarrow -z \frac{d}{dz} X(z)$$

* Find the z-T of $x(n) = n a^n u(n)$.

Sol:-

$$x(n) = n a^n u(n)$$

$$-z \frac{d}{dz} \left[\frac{z}{(z-a)} \right] = \left[\frac{a z}{(z-a)^2} \right]$$

$$a^n u(n) \leftrightarrow \frac{1}{(1-a z^{-1})}$$

$$n a^n u(n) \leftrightarrow -z \frac{d}{dz} \left[\frac{1}{(1-a z^{-1})} \right]$$

$$\leftrightarrow \frac{a z}{(z-a)^2}$$

** Find the IZT of $X(z) = \log(1+a z^{-1})$ $|z| > a$

$$X(z) = \log(1+a z^{-1})$$

Sol:-

$$\frac{d}{dz} X(z) = \frac{1}{1+a z^{-1}}$$

$$n x(n) \leftrightarrow -z \frac{d}{dz} X(z)$$

$$\leftrightarrow -z \frac{d}{dz} [\log(1+a z^{-1})]$$

$$\leftrightarrow -z \frac{1}{(1+az^{-1})} \times -az^{-2}$$

$$\left(\cancel{z} \frac{\cancel{az^{-1}} \times \cancel{z}}{\cancel{z+a}} = \frac{\cancel{a}}{\cancel{z+a}} \right)$$

$$x(z) = \frac{(-a)^n}{n} u(n-1)$$

$$\leftrightarrow -z \left[\frac{-az^{-2}}{1+az^{-1}} \right]$$

$$\leftrightarrow a \frac{z^{-1}}{(1+az^{-1})}$$

$$+ n x(n) \leftrightarrow \frac{a z^{-1}}{[1 - (-az^{-1})]}$$

$$+ n x(n) = a (-a)^{n-1} u(n-1)$$

$$x(n) = \frac{-(-a)^n}{n} u(n-1)$$

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$$* x(z) = \log \frac{1}{[1-az^{-1}]} ; |z| < a$$

$$x(z) = \log \frac{1}{1-az^{-1}}$$

$$= -\log(1-az^{-1})$$

$$n x(n) \leftrightarrow z \frac{d}{dz} [\log(1-az^{-1})]$$

$$\leftrightarrow z \frac{+az^{-2}}{[1-az^{-1}]}$$

$$n x(n) \leftrightarrow \frac{z(1-z^{-1})}{(1-az^{-1})}$$

$$= a \left[-a^{n-1} u(n-1) \right]$$

$$x(n) = -\left(\frac{a^n}{n}\right) u(n-1)$$

* for the system defined as a difference equation $y(n) = \frac{7}{4}y(n-1) - \frac{1}{8}y(n-2) + x(n)$; find the transfer function of its impulse response and find the system response to a unit-step input.

Sol: $y(n) = \frac{7}{4}y(n-1) - \frac{1}{8}y(n-2) + x(n)$

Apply Z.T on both sides.

$$Y(z) = \frac{7}{4} z^{-1} Y(z) - \frac{1}{8} z^{-2} Y(z) + X(z)$$

$$Y(z) \left[1 - \frac{7}{4} z^{-1} + \frac{1}{8} z^{-2} \right] = X(z)$$

$$(a) \quad \frac{Y(z)}{X(z)} = H(z) = \frac{1}{1 + \frac{1}{4} z^{-1} - \frac{7}{8} z^{-2}}$$

$$(b) \quad H(e^{j\omega}) = \frac{1}{1 - \frac{7}{4} e^{-j\omega} + \frac{1}{8} e^{-2j\omega}} \quad [z = e^{j\omega}]$$

(c) inverse of $H(z)$

$$H(z) = \frac{1}{1 - \frac{7}{4} z^{-1} + \frac{1}{8} z^{-2}}$$

$$x_1 = \frac{1}{2}, x_2 = \frac{1}{4}$$

$$= \frac{1}{\left(1 - \frac{1}{2} z^{-1}\right) \left(1 - \frac{1}{4} z^{-1}\right)}$$

$$\boxed{h(n) = 2\left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{4}\right)^n u(n), \quad \checkmark}$$

$$(d) \quad Y(z) = \frac{X(z)}{\left[1 - \frac{1}{2}z^{-1}\right] \left[1 - \frac{1}{4}z^{-1}\right]}$$

$$\text{If } x(n) = u(n)$$

$$X(z) = \frac{1}{[1 - z^{-1}]}$$

$$X(z) = \frac{1}{(1 - z^{-1})(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{4}z^{-1})}$$

$$= \frac{A}{1 - z^{-1}} + \frac{B}{1 - \frac{1}{2}z^{-1}} + \frac{C}{1 - \frac{1}{4}z^{-1}}$$

$$= \frac{8}{3(1 - z^{-1})} + \frac{-2}{1 - \frac{1}{2}z^{-1}} + \frac{1}{3(1 - \frac{1}{4}z^{-1})}$$

$$= \frac{8}{3} u(n) + \left[-2 \left(\frac{1}{2}\right)^n u(n)\right] + \frac{1}{3} \left(\frac{1}{4}\right)^n u(n)$$

$$\frac{1}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{4}z^{-1})}$$

$$\frac{1(2)}{1 - \frac{1}{2}z^{-1}}$$

$$1 - \frac{1}{2}z^{-1}$$

$$A = \frac{2}{3}$$

$$B = \frac{1}{3}$$

$$\frac{1}{1 - \frac{1}{4}z^{-1}}$$

$$C = \frac{1}{3}$$

* A system is defined as a difference equation with initial conditions as shown below

$$3y(n) - 4y(n-1) + y(n-2) = x(n).$$

where $x(n) = (\frac{1}{2})^n u(n)$ & $y(-1) = 1, y(-2) = 2$. Then the system response $y(n]$ is

Sol/1

$$3y(n) - 4y(n-1) + y(n-2) = x(n)$$

Z-T on b.s

$$3[y(z) + y(0)] - 4[z^{-1}y(z) + y(-1)] + [z^{-2}y(z) + z^{-1}y(-1) + y(-2)] = \frac{1}{(1 - \frac{1}{2}z^{-1})}$$

$$3y(z) - 4z^{-1}y(z) + z^{-2}y(z) - 4 + z^{-1} + 2 = \frac{1}{(1 - \frac{1}{2}z^{-1})}$$

$$y(z) \neq [3 - 4z^{-1} + z^{-2}] - 2 + z^{-1} = \frac{1}{1 - \frac{1}{2}z^{-1}}$$

$$y(z) = \frac{1}{(1 - \frac{1}{2}z^{-1})} + 2 - z^{-1}$$

$$\neq \frac{z - 4z^{-1} + z^{-2}}{3 - 4z^{-1} + z^{-2}}$$

$$= \frac{1}{(1 - \frac{1}{2}z^{-1})(3 - 4z^{-1} + z^{-2})} + 2 - z^{-1}$$

$$= \frac{1}{3 - 4z^{-1} + z^{-2} - \frac{3}{2}z^{-1} + 2z^{-1} - \frac{1}{2}z^{-3}} + 2 - z^{-1}$$

$$= \frac{1}{3 - \frac{4}{2} + \frac{1}{2^2} - \frac{3}{2^2} + \frac{2}{2} - \frac{1}{2^3}} + 2 - z^{-1}$$

$$\neq \frac{3z^2 - 4z + 1/2}{3z^2 - 4z + 1/2}$$

$$= \frac{z(3z^2 - 3z + \frac{1}{2})}{3(z-1)(z-\frac{1}{2})(z-\frac{1}{3})}$$

$$\frac{Y(z)}{z} = \frac{A}{z-1} + \frac{B}{z-\frac{1}{2}} + \frac{C}{z-\frac{1}{3}}$$

$$A = \frac{1}{2}, \quad B = -1, \quad C = \frac{1}{2}$$

$$Y(z) = \frac{1}{2} \cdot \frac{z}{z-1} - \left(\frac{z}{z-\frac{1}{2}} \right) + \frac{1}{2} \cdot \frac{z}{(z-\frac{1}{3})}$$

$$y(n) = \frac{1}{2}u(n) - \left(\frac{1}{2}\right)^n u(n) + \frac{1}{2} \left(\frac{1}{3}\right)^n u(n)$$

find the I.Z.T by convolution method

$$X(z) = \frac{z^2}{(z-2)(z-3)}$$

$$X(z) = \left[\frac{z}{z-2} \right] \times \left[\frac{z}{z-3} \right]$$

inverse Z.T by conv.

$$x(n) = 2^n u(n) * 3^n u(n)$$

$$= \sum_{k=-\infty}^{\infty} 2^k u(k) 3^{(n-k)} u(n-k)$$

$$= \sum_{k=0}^n 2^k 3^{n-k}$$

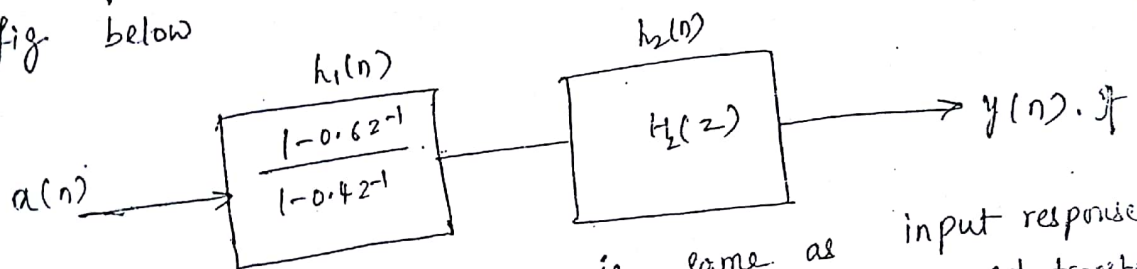
$$= 3^n \cdot \sum_{k=0}^n \left[2^k 3^{-k} \right]$$

$$= 3^n \sum_{k=0}^n \left(\frac{2}{3} \right)^k$$

$$= z^n \frac{(1 - (\frac{2}{3})^n)}{1 - \frac{2}{3}}$$

$$= z^{n+1} u(n) - 2^{n+1} u(n)$$

* Two systems are connected in cascade as shown in the fig below



the output response $y(n)$ is same as $x(n)$ with a unit delay then find the 2nd transfer function $H_2(z)$.

Sol:-

$$y(n) = x(n) * h_1(n) * h_2(n)$$

Z.T on b.s

$$Y(z) = X(z) \times H_1(z) \times H_2(z) \quad \text{--- (1)}$$

$$y(n) = x(n-1)$$

$$Y(z) = z^{-1} X(z) \quad \text{--- (2)}$$

$$X(z) = X(z) H_1(z) \cdot H_2(z)$$

$$H_2(z) = z^{-1} \left(\frac{1 - 0.4z^{-1}}{1 - 0.6z^{-1}} \right)$$

* Discrete system characterization using z-transforms

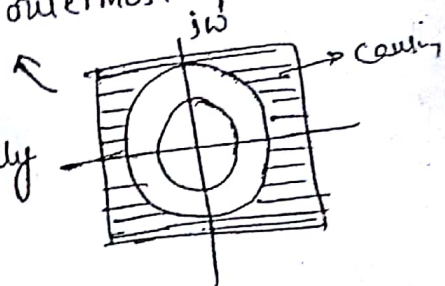
(1) Causality of the system response $H(z)$ is defined as $\frac{N(z)}{D(z)}$ ($H(z) = \frac{N(z)}{D(z)}$) then the system to be causal the order of the numerator cannot be greater than the order of the denominator.

Eg: (1) $H(z) = \frac{(z+1)}{(z^2+3z+5)} \rightarrow \text{causal}$

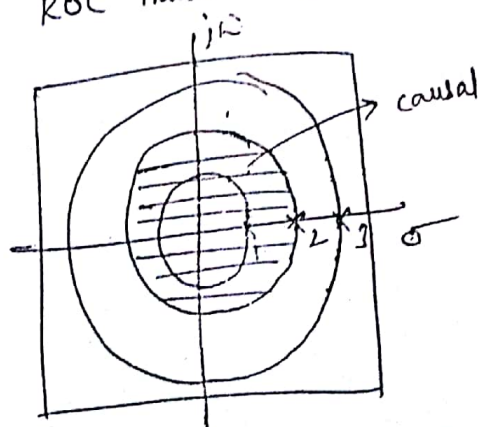
(2) $H(z) = \frac{7z^3 - 2z^2 + z + 1}{z^2 + 3z - 1 + 2} \rightarrow \text{causal}$

(3) $H(z) = \frac{z^2 + 2z + 3}{1 - 2z} \rightarrow \text{Non-causal}$

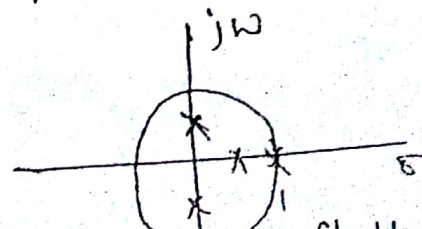
(2) for the system response $H(z)$ to be only causal then its ROC must be the region outside the outermost pole.



* Stability for the system response $H(z)$ to be only stable then the ROC must include the unit circle.



* Both stable and causal for the system response to be both stable and causal all the poles must lie inside the unit circle.



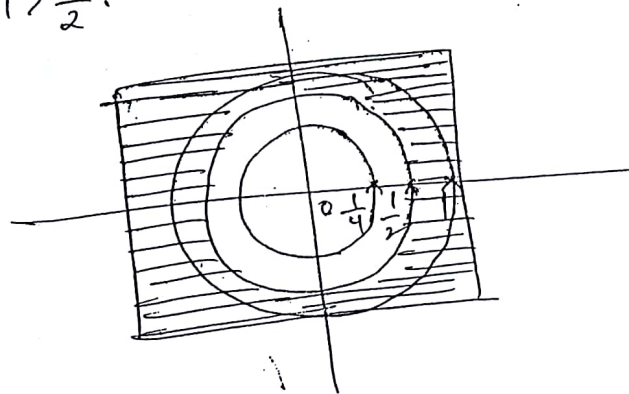
Q. For the given system response $r(z)$...
 for which the system will be only stable, only
 causal both stable and causal $H(z) = \frac{z-z^{-1}}{\left(1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right)}$

Soln

$$H(z) = \frac{z-z^{-1}}{\left(1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right)}$$

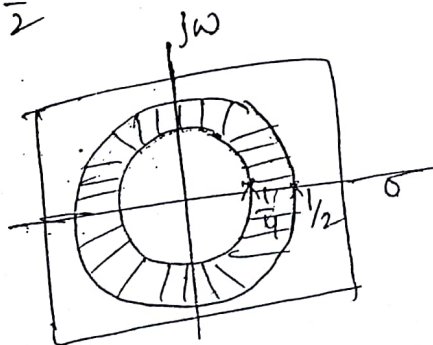
$$= \frac{z-z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)}$$

(a) $|z| > \frac{1}{2}$



- (i) system is stable
 (ii) system is causal

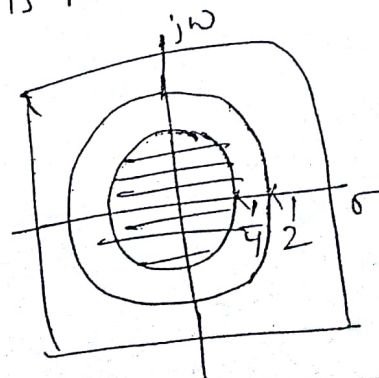
(b) $\frac{1}{4} < |z| < \frac{1}{2}$



- (i) system is neither stable nor causal

(c) $|z| < \frac{1}{4}$

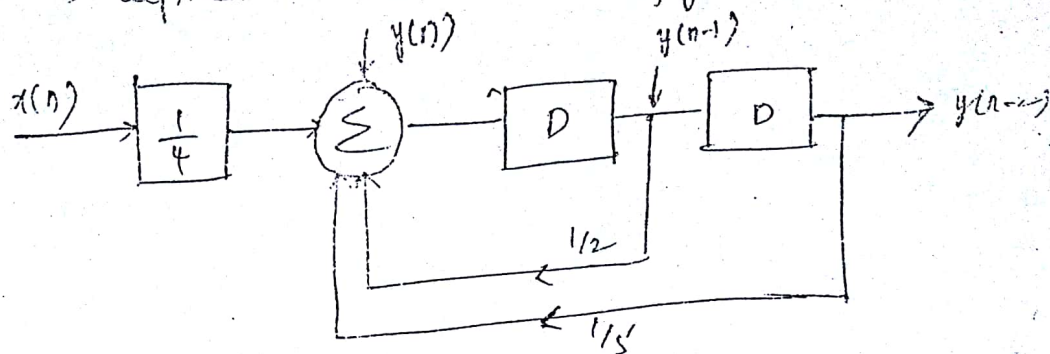
system is neither stable
 nor causal



conditions
by

$$+ \frac{1}{8} z^{-2}$$

Q. A system is defined as shown in the fig below



(a) LTI or LTV

(b) causal or non-causal

(c) stable, both stable & causal, neither stable nor causal

$$y[n] = \frac{1}{4} x[n] + \frac{1}{2} y[n-1] + \frac{1}{5} y[n-2]$$

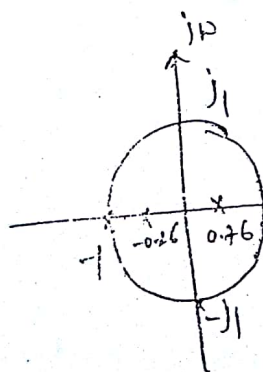
$$a) y[n] - \frac{1}{2} y[n-1] - \frac{1}{5} y[n-2] = \frac{1}{4} x[n] \rightarrow \text{LTI}$$

(b) Applying Z-T on b.s

$$Y(z) - \frac{1}{2} z^{-1} Y(z) - \frac{1}{5} z^{-2} Y(z) = \frac{1}{4} X(z)$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1}{4 \left(1 - \frac{1}{2} z^{-1} - \frac{1}{5} z^{-2} \right)}$$

$$0.76, -0.26$$



system is
both stable &
causal