

Unit-I

Introduction to DSP

Signal:- Anything that carries some information is called Signal. (OR)

It is a physical quantity that varies with time, Space and so on.

For ex:- Voice Signal, Speech Signal, ECG Signal.

There are basically 2 types of Signal.

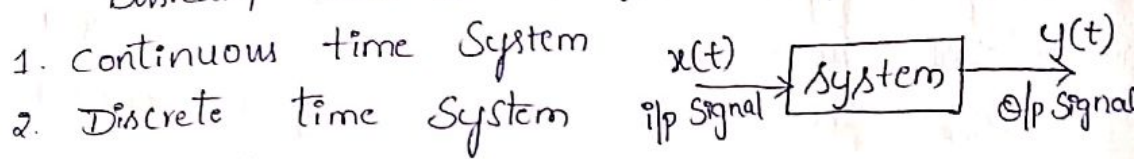
1. Continuous time Signal
2. Discrete time Signal

System:- Interconnection of components

It is a physical device that generates response or O/P for a given i/p.

Basically there are 2 types of Systems.

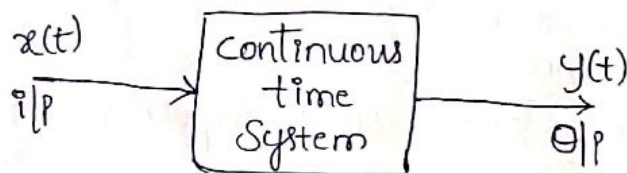
1. Continuous time System
2. Discrete time System



Continuous time System

Continuous time System is one it can operate on continuous time signal and produces continuous time signal as O/P.

Let $x(t)$ & $y(t)$ represent the i/p and O/p of the continuous time system.



Discrete time System:- The system is defined as it can operate on discrete time and produces discrete time signal as the O/P.

Let $x(n)$ and $y(n)$ are the i/p & O/p of the system.

$$y(n) = T[x(n)]$$

Continuous time Signal:- The signals that are defined for every instant of time are known as continuous time signals. It can be represented as $x(t)$.

Discrete time signals:- The signals that are defined for discrete instant of time are known as discrete time signals. It can be represented as $x(n)$.

These are discrete in time, continuous in amplitude.

Digital signals:- These signals are quantized in amplitude and discrete in time are known as digital signals.

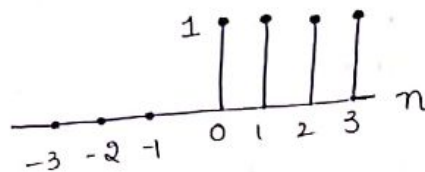
Elementary discrete time sequences

1. Unit step Sequence
2. Unit ramp Sequence
3. Unit impulse Sequence
4. Exponential
5. Sinusoidal
6. complex Exponential

Unit step Sequence

The discrete time unit step sequence can be defined as $u(n) = 1$ for $n \geq 0$
 $= 0$ for $n < 0$

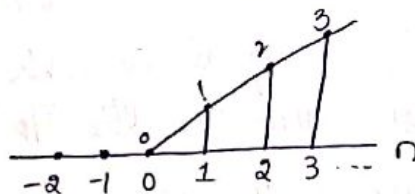
The graphical representation of $u(n)$ may be represented as follows.



Unit ramp Sequence

The discrete time unit ramp sequence can be represented as $r(n) = n$ for $n \geq 0$
 $= 0$ for $n < 0$

The graphical representation of $r(n)$ may be represented as

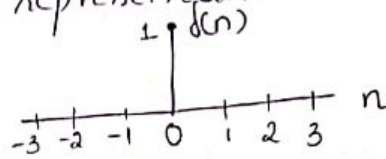


Unit Impulse Sequence

The discrete time Unit Impulse Sequence can be defined as

$$\delta(n) = 1 \text{ for } n=0 \\ = 0 \text{ for } n \neq 0$$

The graphical representation of $\delta(n)$ can be represented as



Exponential Sequence

The Exponential Signal can be defined as $x(n) = a^n$ for all 'n'. This exponential Sequence can be dependend on the value of a i.e., $a > 1$ (or) $0 < a < 1$ (or) $a < 0$

Sinusoidal Sequence

The discrete time Sinusoidal Signal can be defined as $x(n) = A \cos(\omega_0 n + \phi)$. Where ω_0 is frequency measured in rad / Samples.

$\phi \rightarrow$ phase (radians)

By using Euler's identity, the above expression can be written as

$$x(n) = \frac{A}{2} e^{j\phi} e^{jn\omega_0} + \frac{A}{2} e^{-j\phi} e^{-jn\omega_0}$$

Complex Exponential Sequence

The complex exponential Sequence can be defined as $x(n) = a^n e^{j(\omega_0 n + \phi)}$

$$= a^n [\cos(\omega_0 n + \phi) + j \sin(\omega_0 n + \phi)]$$

Representation of discrete time Signals

There are 4 different ways to represent the discrete time Signals.

1. Graphical representation
2. Functional representation
3. Tabular representation
4. Sequence representation

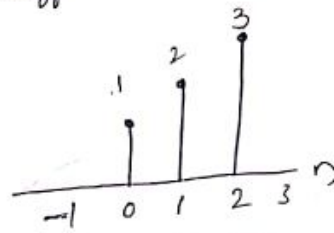
Graphical representation:- Let us consider a discrete time signal $x(n)$ having different values i.e.,

$$x(-1) = 0$$

$$x(0) = 1$$

$$x(1) = 2$$

$$x(2) = 3$$



Functional representation:- The discrete time signal $x(n)$ can be represented in the form of functional representation.

$$x(n) = \begin{cases} -2 & \text{for } n=2 \\ -1 & \text{for } n=1 \\ 0 & \text{for } n=0 \\ 1 & \text{for } n=-1 \\ 2 & \text{for } n=-2 \end{cases}$$

Tabular representation:- The discrete time signal $x(n)$ can be represented in tabular form.

n	-1	-2	0	1	2
$x(n)$	0	1	2	3	4

Sequence representation:- The finite duration sequence $x(n)$ the time origin for $n=0$ can be represented by an arrow $\uparrow$.

For ex:- $x(n) = \{1, 2, -1, -2\}$

An infinite duration sequence $x(n)$ the time origin can be represented as $x(n) = \{ \dots -1, 0, 1, 2, 3, \dots \}$

Classification of discrete time signals

1. Energy and power signals
2. Periodic and aperiodic signals
3. Symmetric and asymmetric signals
4. Casual and non-casual signals

Energy and power Signals

The Energy Signal can be defined as

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

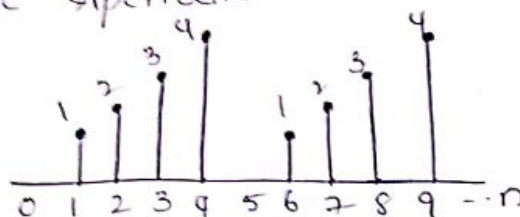
The power Signal can be defined as

$$P = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x(n)|^2$$

periodic and Aperiodic Signals

A Signal is said to be periodic must satisfies the condition $x(n) = x(n+N)$. Otherwise the Signal is said to be aperiodic.

for $N=4$



Symmetric and Asymmetric Signals

A Signal is said to be Symmetric Signals or even Symmetry must satisfies the condition $x(n) = x(-n)$

for ex:- A $\cos \omega n$

A Signal is said to be Odd Symmetry or Asymmetric Signal must satisfies the condition is

$$-x(n) = x(-n)$$

(or)

$$x(n) = -x(-n)$$

A discrete time Signal $x(n)$ can be represented as

$$x(n) = x_e(n) + x_o(n)$$

In the above eqⁿ n is replaced by ' $-n$ ' then

$$x(-n) = x_e(-n) + x_o(-n)$$

According to definition $x(-n) = x_e(n) - x_o(n)$

Combining $x(n)$ & $x(-n)$ then we can get

$$x(n) = x_e(n) + x_o(n)$$

$$x(-n) = x_e(n) - x_o(n)$$

$$\hline x(n) + x(-n) = 2x_e(n)$$

$$x_e(n) = \frac{x(n) + x(-n)}{2}$$

Similarly Subtracting $x(n)$ & $x(-n)$ then we can get

$$x(n) = x_e(n) + x_o(n)$$

$$x(-n) = x_e(n) - x_o(n)$$

$$x(n) - x(-n) = 2x_o(n)$$

$$x_o(n) = \frac{x(n) - x(-n)}{2}$$

Casual and non-casual Signals

A Signal is said to be casual Signal if its value '0' for $n < 0$. Otherwise the Signal is said to be non-casual [for $n \geq 0$].

Ex:- for casual Signal is $x(n) = a^n u(n)$
for non-casual Signal $x(n) = a^n u(-n+1)$

Classification of discrete time Systems

- 1) static and dynamic Systems
- 2) casual and non-casual Systems
- 3) Linear and non-linear Systems
- 4) Time invariant and Time variant Systems
- 5) FIR and IIR Systems
- 6) Stable and unstable Systems

Static and dynamic Systems

A discrete time System is called static if its output at any instant of time depends on the input sample at the same time but not on past and future samples of the input, in any other case the System is said to be dynamic System.

The Examples are

$$y(n) = a x(n) \text{ for static System}$$

$$y(n) = x(n-1) + x(n+2) \text{ for dynamic System.}$$

Casual and non-casual Systems

A System is said to be casual if the Output of the system at any time depends only at present and past inputs but does not depend on future inputs. It can be represented as

$$y(n) = f(x(n), x(n-1), x(n-2), \dots)$$

If the Output of a system depends on future inputs the system is said to be non-casual. It can be represented as

$$y(n) = x[2n] + x(n+1).$$

Linear and non-linear Systems

A system that satisfies Superposition principles is said to be a linear system. The Superposition principle states that the response of the system to a weighted sum of signals should be equal to the corresponding weighted sum of the outputs of the systems to each of the individual input signals.

A system is linear if and only if

$$T[ax_1(n) + bx_2(n)] = aT[x_1(n)] + bT[x_2(n)]$$

Here a and b are arbitrary constants.

A system that does not satisfy the Superposition principle is called non-linear system.

Time invariant and Time variant Systems

A system is said to be time invariant or shift invariant if the characteristics of the system does not change with time.

For a time invariant system if $y(n)$ is the response of the system to the input $x(n)$ then response of the system to the input $x(n-k)$ is $y(n-k)$.

$$y(n-k) = T[x(n-k)]$$

$y(n-k) \neq T[x(n-k)]$ then the system is said to be time variant.

A linear time invariant System (LTI) satisfies linearities in the time invariant properties.

FIR and IIR Systems

If the impulse response of the system is of finite duration, then the system is called FIR System.

Ex:- $h(n) = \begin{cases} 1 & \text{for } n = -1 \\ 2 & \text{for } n = 0 \\ 3 & \text{for } n = 1 \end{cases}$

A System has an impulse response for infinite duration is called IIR System.

Ex:- $h(n) = a^n u(n)$

Stable and unstable Systems

An LTI System is stable if it produces a bounded O/p sequence for every bounded i/p sequence.

If for some bounded i/p sequence $x(n)$ the O/p is unbounded the system is called unstable. The necessary and sufficient condition for stability is

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Problems

1] Test the stability of the system whose impulse response $h(n) = \left(\frac{1}{2}\right)^n u(n)$.

Sol:- To test the stability the given condition is

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Given data

$$h(n) = \left(\frac{1}{2}\right)^n u(n)$$

$$= \sum_{n=-\infty}^{\infty} \left[\left(\frac{1}{2}\right)^n u(n) \right]$$

$$\begin{aligned}
&= \sum_{n=-\infty}^0 \left[\left(\frac{1}{2}\right)^n u(n) \right] + \sum_{n=0}^{\infty} \left[\left(\frac{1}{2}\right)^n u(n) \right] \\
&= 0 + \sum_{n=0}^{\infty} \left[\left(\frac{1}{2}\right)^n u(n) \right] \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n u(n) \quad \left[\because u(n) = 1 \text{ for } n \geq 0 \right] \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \\
&= 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots + \infty \\
&= 1 + x + x^2 + x^3 + \dots + \infty = \frac{1}{1-x} \\
&= \frac{1}{1-1/2} = 2 < \infty
\end{aligned}$$

$\therefore$ The given system is stable.

2] Test the given system for time invariant or not
 $y(n) = n x(n)$.

Sol:- Given data

$$y(n) = n x(n)$$

$$y(n-k) \neq T[x(n-k)]$$

$\therefore$ The given system is time variant.

3] Determine whether the following system given by
 $y(n) = \log_{10} [x(n)]$ is casual or not.

Sol:- Given data

$$y(n) = \log_{10} [x(n)]$$

for $n=0$

$$y(0) = \log_{10} x(0)$$

for $n=-1$

$$y(-1) = \log_{10} x(-1)$$

$\therefore$ The given system is casual system.

4] Determine wheather the system defined by
 $y(n) = x(-n^2 - 2)$ is time variant or not.

Sol:- Given data

$$y(n) = x(-n^2 - 2)$$

$$y(n-k) = x[-(n-k)^2 - 2]$$

$\therefore$ The given system is time variant.

5] Test the following systems for time invariant

$$y(n) = n x^*(n).$$

Sol:- Given data

$$y(n) = n x^*(n)$$

$$y(n-k) \neq T[x(n-k)]$$

$\therefore$ The given system is time variant.

6] Test wheather the following signal is periodic or not
 If periodic find the fundamental period $\sin \sqrt{2} \pi t$

Sol:- Given $\sin \sqrt{2} \pi t$

periodic signal $\Rightarrow x(t) = x[t+T]$

$$\text{Let } x(t) = \sin \sqrt{2} \pi t = \sin \omega_0 t$$

$$\omega_0 = \sqrt{2} \pi$$

$$2\pi f = \sqrt{2} \pi$$

$$\frac{2\pi}{T} = \sqrt{2} \pi$$

$$T = \frac{2}{\sqrt{2}}$$

$$T = \sqrt{2}$$

7] Test wheather the following signal is periodic or not.
 If periodic find the fundamental period

$$\sin 20\pi t + \sin 5\pi t.$$

Sol:- Given Data

$$x(t) = \sin 20\pi t + \sin 5\pi t = \sin \omega_1 t + \sin \omega_2 t$$

$$\omega_1 = 20\pi$$

$$\frac{2\pi}{T_1} = 20\pi$$

$$T_1 = \frac{2}{10} = \frac{1}{5} = 0.2$$

$$\omega_2 = 5\pi$$

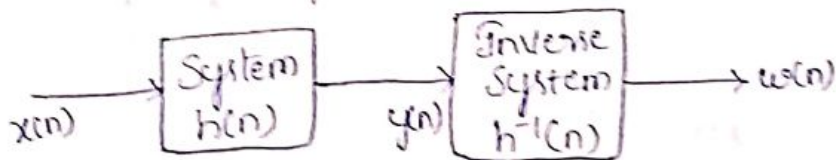
$$\frac{2\pi}{T_2} = 5\pi$$

$$T_2 = \frac{2}{5} = 0.4$$

$$\therefore T = \frac{T_1}{T_2} = \frac{0.1}{0.4} = 0.25$$

Invertability (or) Inverse System

- A System is said to be invertible we can generate the i/p signal $x(n)$ from the o/p of a system $y(n)$.
- It can be generated by using cascading of inverse system with the original system.
- It can be represented as



$$\begin{aligned} w(n) &= y(n) * h^{-1}(n) \\ &= [x(n) * h(n)] * h^{-1}(n) \\ &= x(n) * h(n) * h^{-1}(n) \\ &= x(n) \end{aligned}$$

* Determine the convolution of two sequences
 $x(n) = \{3, 2, 1, 2\}$ $h(n) = \{1, 2, 1, 2\}$

Sol:- Given Sequences

$$x(k) = \{3, 2, 1, 2\}$$

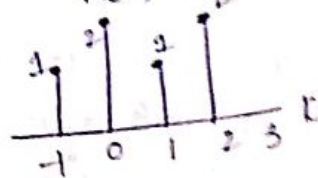
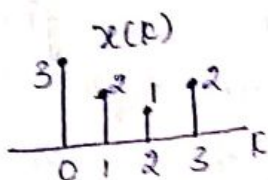
$$h(k) = \{1, 2, 1, 2\}$$

Length of the Sequence
 $x(k) = 4 = N_1$

Length of the Sequence
 $h(k) = 4 = N_2$

$$\text{Total O/p Sequence} = N_1 + N_2 - 1 = 4 + 4 - 1 = 8 - 1 = 7$$

$$n_1 = 0 \text{ to } n_2 = 3 \Rightarrow n_1 + n_2 - 0 - 1 = -1$$

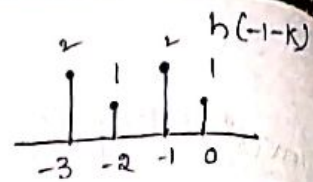


$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

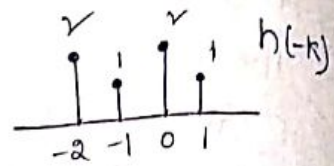
$$y(n) = \sum_{k=-1}^5 x(k) h(n-k)$$

$$[k = -1, 0, 1, 2, 3, 4, 5]$$

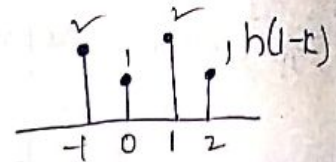
$$\text{for } n=-1 \Rightarrow y(-1) = \sum x(k) h(-1-k) \\ = 1 \times 3 = 3$$



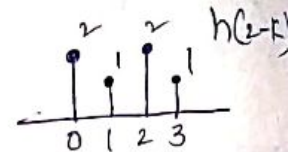
$$y(0) = \sum x(k) h(-k) \\ = (3 \times 2) + (1 \times 2) \\ = 6 + 2 = 8$$



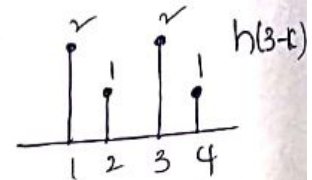
$$y(1) = \sum x(k) h(1-k) \\ = (1 \times 3) + (2 \times 2) + (1 \times 1) \\ = 3 + 4 + 1 = 8$$



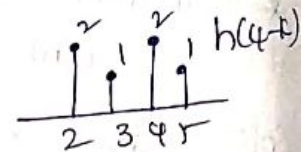
$$y(2) = \sum x(k) h(2-k) \\ = (3 \times 2) + (1 \times 2) + (2 \times 1) + (1 \times 2) \\ = 12$$



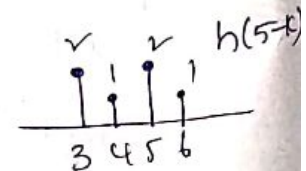
$$y(3) = \sum x(k) h(3-k) \\ = (2 \times 2) + (1 \times 1) + (2 \times 2) \\ = 9$$



$$y(4) = \sum x(k) h(4-k) \\ = 2 + 2 = 4$$



$$y(5) = \sum x(k) h(5-k) \\ = 4$$



$$y(n) = \{3, 8, 8, 12, 9, 4, 4\}$$

Convolution

- For a linear time invariant system with the input sequence $x(n)$ and impulse response $h(n)$ are given we can find the o/p $y(n)$ by using the equation

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k) \text{ which is known as convolution sum and can be represented as } y(n) = x(n) * h(n).$$

where $*$ represents the convolution operation.

The convolution sum of two sequences can be found by using the following steps.

Step 1:- Choose an initial value of n , the starting time for evaluating the O/p Sequence $y(n)$. If $x(n)$ starts at $n=n_1$ and $h(n)$ starts at $n=n_2$ then $n=n_1+n_2$ is considerable choice.

Step 2:- Express both Sequences in terms of the index k .

Step 3:- Fold $h(k)$ about $k=0$ to obtain $h(-k)$ and shift by ' n ' to the right if n is positive and left if n is negative to obtain $h(n-k)$.

Step 4:- Multiply the two Sequences $x(k)$ and $h(n-k)$ element by element and sum of the products to get $y(n)$.

Step 5:- Increment the index n , shift the sequence $h(n-k)$ to right by one sample and do step 4.

Step 6:- Repeat Step 5 until the sum of products is zero for all the remaining values of ' n '.

Properties of convolution

1. Commutative law

$$x(n) * h(n) = h(n) * x(n)$$

2. Associative law

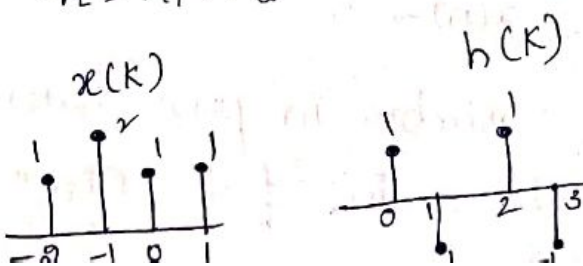
$$(x(n) * h_1(n)) * h_2(n) = x(n) * [h_1(n) * h_2(n)]$$

3. Distributive law

$$x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$$

* Find the convolution of two Sequences $x(n) = \{1, 2, 1, 1\}$
 $h(n) = \{1, -1, 1, -1\}$

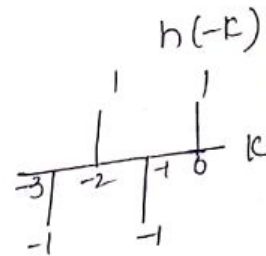
Sol:- $n_1 = -2$ and $n_2 = 0$
 $n = n_1 + n_2 = -2 + 0 = -2$



$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

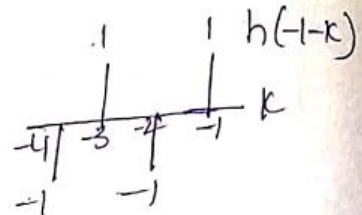
$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

for $n=0$:- $y(0) = \sum x(k) h(-k)$
 $= 1 - 2 + 1 = 0$



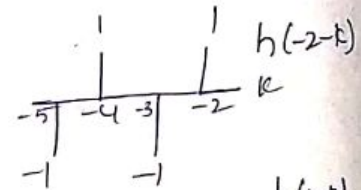
$$y(-1) = \sum x(k) h(-1-k)$$

$$= -1 + 2 = 1$$



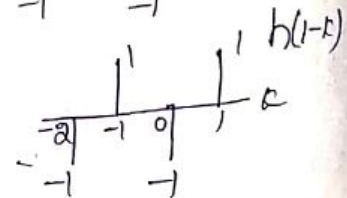
$$y(-2) = \sum x(k) h(-2-k)$$

$$= 1$$



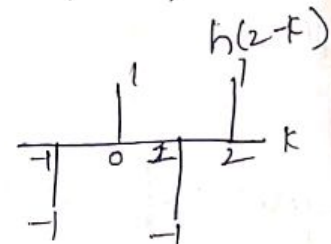
$$y(1) = \sum x(k) h(1-k)$$

$$= -1 + 2 - 1 = 0$$



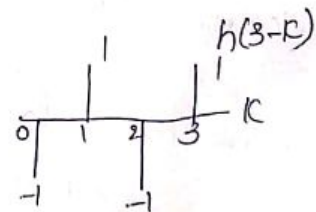
$$y(2) = \sum x(k) h(2-k)$$

$$= -2 + 1 - 1 = -2$$



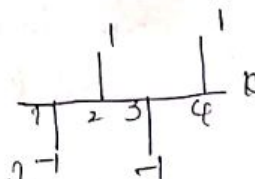
$$y(3) = \sum x(k) h(3-k)$$

$$= -1 + 1 - 1 = -1$$



$$y(4) = \sum x(k) h(4-k)$$

$$= 0$$



$$y(n) = \{1, 1, 0, 1, -2, 0, -1\}$$

Z-Transform

The Z-transform of a discrete time signal $x(n)$ is defined as $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$.

Where z is a complex variable in polar form $z = re^{j\omega}$. Where r is the radius of the circle.

The z -value Substitute in the above equation

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (re^{j\omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) r^{-n} \cdot e^{-j\omega n}$$

- The value of r is consider to be 1 then

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$\therefore$ The z -transform is converted into Fourier transform whenever $r=1$.

- This is also called as two sided z -transform.

- The discrete time signal $x(n)$ is considered to be a casual signal then.

$$X_T(z) = \sum_{n=0}^{\infty} x(n) z^{-n}, \text{ this is called}$$

One Sided z -transform.

Region of convergence (ROC)

The set of all values of z for which $x(z)$ gets finite value is called ROC of $x(z)$.

Right hand Sequence:- A sequence is said to be right hand sequence for which $x(n)=0$ for $n < n_0$.

Where n_0 may be +ve or -ve but finite. The ROC of right hand sequence will be entire z -plane except $z=0$.

For Ex:- $x(n) = \{1, 0, -2, 3, 4\}$

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= x(0) z^0 + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + x(4) z^{-4}$$

$$= 1 - 2z^{-2} + 3z^{-3} + 4z^{-4}$$

The ROC of $x(z)$ will be entire z -plane except $z=0$

Left hand Sequence:- A Sequence is said to be left hand Sequence for which $x(n) = 0$ for $n \geq n_0$. Where n_0 may be +ve or -ve but finite value.
 - The ROC of left hand Sequence will be Entire z -plane except $z = \infty$.

For Ex:- $x(n) = \{-3, -2, -1, 0\}$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= x(-3) z^3 + x(-2) z^2 + x(-1) z^1 + x(0) z^0$$

$$= -3z^3 - 2z^2 - z$$

The ROC of $x(z)$ will be Entire z -plane except $z = \infty$.
Two Sided Sequence:- A Signal that has finite duration on both left and right hand sides is known as two side Sequence for such type of Sequence the ROC is Entire z -plane except $z = 0$ and $z = \infty$.

For Ex:- $x(n) = \{2, -1, 3, 2, 1, 0, 2, 3, -1\}$. Find $x(z)$.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= x(-4) z^4 + x(-3) z^3 + x(-2) z^2 + x(-1) z^1 + x(0) z^0 + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + x(4) z^{-4}$$

$$= 2z^4 - z^3 + 3z^2 + 2z + 1 + 2z^{-2} + 3z^{-3} - z^{-4}$$

$$X(z) = 2z^4 - z^3 + 3z^2 + 2z + 1 + 2z^{-2} + 3z^{-3} - z^{-4}$$

- The ROC of $x(z)$ will be Entire z -plane except $z = 0$ and $z = \infty$.

properties of ROC

- 1] The ROC is a ring/disk in z -plane centered at origin.
- 2] The ROC cannot contain any poles.
- 3] The ROC of a casual Sequence will be Entire z -plane except $z=0$.
- 4] The ROC of a non-casual Sequence will be Entire z -plane except $z=\infty$.
- 5] The ROC of finite duration two Sided Sequence will be Entire z -plane except $z=0$ and $z=\infty$.
- 6] The ROC of in-finite duration two Sided Sequence will be ring in the z -plane.
- 7] The ROC of LTI stable System contain unit circle.
- 8] The ROC must be a connected region.

properties of z -transform

1] Linearity property:-

$$\text{If } z[x_1(n)] = x_1(z)$$

$$z[x_2(n)] = x_2(z)$$

$$\text{then } z[ax_1(n) + bx_2(n)] = ax_1(z) + bx_2(z)$$

$$\begin{aligned} \text{proof:- } z[ax_1(n) + bx_2(n)] &= \sum_{n=-\infty}^{\infty} [ax_1(n) + bx_2(n)] z^{-n} \\ &= \sum_{n=-\infty}^{\infty} ax_1(n) z^{-n} + \sum_{n=-\infty}^{\infty} bx_2(n) z^{-n} \\ &= a \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + b \sum_{n=-\infty}^{\infty} x_2(n) z^{-n} \end{aligned}$$

$$\boxed{z[ax_1(n) + bx_2(n)] = ax_1(z) + bx_2(z)}$$

2] Time shifting property

$$\text{If } z[x(n)] = x(z)$$

$$\text{then } z[x(n-m)] = z^{-m} x(z)$$

Proof:- $\mathcal{Z}[x(n-m)] = \sum_{n=-\infty}^{\infty} x(n-m) z^{-n}$

Let $n-m = l$

$n = l+m$

$= \sum_{n=-\infty}^{\infty} x(l) z^{-(l+m)}$

$= \sum_{n=-\infty}^{\infty} x(l) \cdot z^{-l} \cdot z^{-m}$

$= z^{-m} \sum_{n=-\infty}^{\infty} x(l) \cdot z^{-l}$

$\boxed{\mathcal{Z}[x(n-m)] = z^{-m} X(z)}$

3) Time Reversal property

If $\mathcal{Z}[x(n)] = X(z)$

then $\mathcal{Z}[x(-n)] = X(z^{-1})$

Proof:- $\mathcal{Z}[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$

Let $-n = l$

$= \sum_{l=-\infty}^{\infty} x(l) z^{-l}$

$= \sum_{l=-\infty}^{\infty} x(l) (z^{-1})^{-l}$

$\boxed{\mathcal{Z}[x(n)] = X(z^{-1})}$

4) Differentiation Property:- If z -transform of $\mathcal{Z}[x(n)] = X(z)$ then

$\mathcal{Z}[n x(n)] = -z \frac{d}{dz} X(z)$

Proof:- $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$\frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot (-n) \cdot z^{-n-1}$

$$\frac{d}{dz} x(z) = \sum_{n=-\infty}^{\infty} nx(n) z^{-n} \left(-\frac{1}{z}\right)$$

Multiplying $-z$ on both sides of eqn.

$$-z \frac{d}{dz} x(z) = -z \sum_{n=-\infty}^{\infty} nx(n) z^{-n} \left(-\frac{1}{z}\right)$$

$$-z \frac{d}{dz} x(z) = \sum_{n=-\infty}^{\infty} nx(n) z^{-n}$$

$$= z [nx(n)]$$

$$z[nx(n)] = -z \frac{d}{dz} x(z)$$

5. Convolution property :-

If $z[x(n)] = X(z)$ and $z[h(n)] = H(z)$ then

$$z[x(n) * h(n)] = X(z)H(z)$$

Proof:- Convolution of two sequences

$$x(n) * h(n) = y(n)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

$$z[x(n) * h(n)] = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x(k)h(n-k) z^{-n}$$

Interchange the order of summation.

$$= \sum_{k=-\infty}^{\infty} x(k) \sum_{n=-\infty}^{\infty} h(n-k) z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x(k) \sum_{l=-\infty}^{\infty} h(l) z^{-l} z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} x(k) z^{-k} \sum_{l=-\infty}^{\infty} h(l) z^{-l}$$

$$= X(z)H(z)$$

6. Initial value theorem:-

If $z[x(n)] = X_+(z)$ then

$$x(0) = \lim_{z \rightarrow \infty} X_+(z) \quad \text{where}$$

$$X_+(z) = \sum_{n=0}^{\infty} x(n)z^{-n} \text{ in this eqn.}$$

Sub. In $z = +\infty$ all the values are vanish except $x(0)$.

7. Final value theorem :-

$$\text{If } z[x(n)] = X_+(z) \text{ then } x(\infty) = \lim_{z \rightarrow 1} (z-1)X_+(z)$$

System Functions :- In general the system is described by a linear constant co-efficient differential equation of the form

$$\sum_{k=0}^N a_k y(n-k)$$

$$= \sum_{k=0}^M b_k x(n-k)$$

$$a_0 y(n) + \sum_{k=1}^N a_k y(n-k) = \sum_{k=0}^M b_k x(n-k)$$

$$\text{Let } a_0 = 1$$

$$y(n) = -\sum_{k=1}^N a_k y(n-k) + \sum_{k=0}^M b_k x(n-k)$$

Taking z -transform on b.s & applying time shifting property we can get

$$Y(z) = -\sum_{k=1}^N a_k Y(z)z^{-k} + \sum_{k=0}^M b_k X(z)z^{-k}$$

$$Y(z) \left[1 + \sum_{k=1}^N a_k z^{-k} \right] = \sum_{k=0}^M b_k X(z) z^{-k}$$

$$\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^M b_k \cdot z^{-k}}{1 + \sum_{k=1}^N a_k \cdot z^{-k}}$$

$$H(z) = \frac{\sum_{k=0}^M b_k \cdot z^{-k}}{1 + \sum_{k=1}^N a_k \cdot z^{-k}}$$

* Here $H(z)$ is known as transfer function (or) system function.

* $Y(z)$ is the z -transform of the o/p sequence $y(n)$ and $X(z)$ is the z -transform of the i/p sequence $x(n)$.

* By using the system function we can generate poles & zeroes. i.e., the zeros of the system function $H(z)$ are the values of ' z ' for which $H(z)=0$

* The poles of the system function are the values of ' z ' for which $H(z)=\infty$.

* The system function contains both poles & zeroes and hence the corresponding system is called pole zero system with ' n ' poles & ' m ' zeroes.

① Find the system function and impulse response of the system describe by the difference eqn $y(n) = \frac{1}{5}y(n-1) + x(n)$

so) $y(n) = \frac{1}{5}y(n-1) + x(n)$

Applying z -transform on b.s

$$Y(z) = \frac{1}{5}Y(z)z^{-1} + X(z)$$

$$Y(z) [1 - \frac{1}{5}z^{-1}] = X(z) \Rightarrow \frac{X(z)}{Y(z)} = \frac{1}{1 - \frac{1}{5}z^{-1}}$$

$$H(z) = \frac{1}{1 - \frac{1}{5}z^{-1}}$$

Applying inverse z -transform on b.s

$$h(n) = (\frac{1}{5})^n u(n)$$

② Find a z -transform of response $x(n) = (\frac{1}{3})^{n-1} u(n-1)$

so) Given data

$$x(n) = (\frac{1}{3})^{n-1} u(n-1)$$

$$x(n) = (Y_3)^n u(n)$$

$$X(z) = \frac{1}{1 - Y_3 z^{-1}}$$

By using time shifting property

$$\begin{aligned} z[(Y_3)^{n-1} u(n-1)] &= z^{-1} X(z) \\ &= z^{-1} \frac{z}{z - \frac{1}{3} Y_3} \end{aligned}$$

$$\boxed{= \frac{1}{z - Y_3}}$$

⑩ Find z-transform of sequence $x(n) = n \cdot a^n u(n)$

sol) Given data,

$$x(n) = n a^n u(n)$$

$$z[a^n u(n)] = \frac{1}{1 - a z^{-1}} = \frac{z}{z - a}$$

By using differentiate property

$$z[n x(n)] = -z \frac{d}{dz} x(z)$$

$$= -z \frac{d}{dz} \left(\frac{z}{z - a} \right)$$

$$= -z \left(\frac{z - a - z}{(z - a)^2} \right)$$

$$\boxed{z[n x(n)] = a z / (z - a)^2}$$

Time Response Analysis of discrete time systems:-

* The difference eqn of nth order discrete time system can be written as

$$\sum_{k=0}^N a_k y(n-k) = \sum_{k=0}^M b_k x(n-k)$$

* If the co-efficient $a_0 \neq 1$ the solution of the difference eqn. can be expressed as some of two parts given by

$$y(n) = y_h(n) + y_p(n)$$

Where $y_h(n)$ is known as homogeneous solution and $y_p(n)$ is called particular solution.

Natural Response (or) Zero i/p Response :-

* The natural response is the solution of the above difference equation with $x(n) = 0$

$\therefore$ For a discrete time system the natural response is the solution of homogeneous equation.

$$\text{i.e., } \sum_{k=0}^N a_k y(n-k) = 0$$

* The solution of this equation is of the form $y_h(n) = \lambda^n$

sub. these value in the above eqn. we can get

$$\sum_{k=0}^N a_k \lambda^{n-k} = 0$$

$$a_0 \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \dots + a_{N-1} \lambda^{n-(N-1)} + a_N \lambda^{n-N} = 0$$

$$\text{let, } a_0 = 1$$

$$\lambda^{n-N} [\lambda^N + a_1 \lambda^{N-1} + \dots + a_{N-1} \lambda + a_N] = 0$$

Which gives,

$$\lambda^N + a_1 \lambda^{N-1} + \dots + a_{N-1} \lambda + a_N = 0.$$

The above eqn. is characteristic eqn. of the system

∴ The n^{th} order characteristic eqn. can be expressed in characterised form as

$$(\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_N) = 0$$

* Where, $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_N$ are called roots of the characteristic equation (or) Eigen values of the system.

* The nature of the roots will be real, imaginary and complex roots.

* If the roots $\lambda_1, \lambda_2, \dots, \lambda_N$ are distinct in that case the general solution of homogeneous eqn.

$$y_h(n) = c_1 \lambda_1^n + c_2 \lambda_2^n + c_3 \lambda_3^n + \dots + c_N \lambda_N^n$$

Where, $c_1, c_2, c_3, \dots, c_N$ are arbitrary constants.

* If the roots are repeated for example if the roots of the characteristic eqn. $\lambda_1 = -2$ & $\lambda_2 = -2$ and $\lambda_3 = 2$ then the solution will be

$$y_h(n) = [c_1 + c_2 n] \lambda_1^n + c_3 \lambda_3^n$$

* If the roots are complex then $\lambda_1 = a + ib$ $\lambda_2 = a - ib$

$$y_h(n) = r [A_1 \cos \omega \theta + A_2 \sin \omega \theta]$$

$$r = \sqrt{a^2 + b^2}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

Where A_1 & A_2 are constants.

*
*
* (P) Find the natural response of the system described by the difference equation $y(n) + 2y(n-1) + y(n-2) = x(n) + x(n-1)$ with initial conditions $y(-1) = y(-2) = 1$

-sol Given diff eqn,

$$y(n) + 2y(n-1) + y(n-2) = x(n) + x(n-1)$$

$$y(n) + 2y(n-1) + y(n-2) = 0$$

$$\lambda^n + 2\lambda^{n-1} + \lambda^{n-2} = 0$$

$$\lambda^{n-2} [\lambda^2 + 2\lambda + 1] = 0$$

$$\lambda^2 + 2\lambda + 1 = 0$$

$$(\lambda + 1)^2 = 0$$

$$\lambda_1 = -1, \lambda_2 = -1$$

$$y(n) = [c_1 + c_2 n] (-1)^n$$

By applying initial conditions

$$y(0) = c_1$$

$$y(1) = (c_1 + c_2) (-1)$$

$$= -c_1 - c_2$$

$$y(n) + 2y(n-1) + y(n-2) = 0$$

$$\text{Sub } y(n) = 0$$

$$y(0) + 2y(0-1) + y(0-2) = 0$$

$$y(0) + 2 + 1 = 0$$

$$y(0) = -3$$

$$y(1) + 2y(0) + y(-1) = 0$$

$$y(1) - 6 + 1 = 0$$

$$y(1) = 5$$

$$\boxed{c_1 = -3}$$

$$5 = -c_1 - c_2$$

$$5 = 3 - c_2$$

$$\boxed{c_2 = -2}$$

$$\boxed{y_h(n) = [-3 + (-2)n] (-1)^n u(n)}$$

① Find the natural response of the system described by the difference equation $y(n) - 4y(n-1) + 4y(n-2) = x(n) - x(n-1)$ with initial conditions $y(-1) = y(-2) = 1$

sol) Given diff. eqn

$$y(n) - 4y(n-1) + 4y(n-2) = x(n) - x(n-1)$$

$$y(n) - 4y(n-1) + 4y(n-2) = 0$$

$$\lambda^n - 4\lambda^{n-1} + 4\lambda^{n-2} = 0$$

$$\lambda^{n-2} [\lambda^2 - 4\lambda + 4] = 0$$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$(\lambda - 2)^2 = 0$$

$$\lambda_1 = 2 ; \lambda_2 = 2$$

$$y(n) = [c_1 + c_2 n](\lambda)^n$$

$$y(n) = [c_1 + c_2 n](2)^n$$

By applying initial conditions

$$y(0) = [c_1](2)^0 = c_1$$

$$y(1) = [c_1 + c_2](2)^1$$

$$= [c_1 + c_2](2)$$

$$y(n) - 4y(n-1) + 4y(n-2) = 0$$

$$y(0) - 4y(-1) + 4y(-2) = 0$$

$$y(0) - 4 + 4 = 0$$

$$y(0) = 0$$

$$y(1) - 4y(0) + 4y(-1) = 0$$

$$y(1) - 0 + 4 = 0$$

$$y(1) = -4$$

$$c_1 = 0$$

$$2(c_1 + c_2) = -4$$

$$0 + 2c_2 = -4$$

$$c_2 = -2$$

$$y_h(n) = [-3 + (-2)^n] (-1)^n u(n)$$

Forced Response :-

The forced Response of the system is obtained by summing the particular solution and homogenous solution and finding the coefficients the homogeneous solution so that the combined response $y_h(n) + y_p(n)$ satisfies the zero initial conditions.

General form of particular solution for several types of inputs as shown in table.

<u>i/p signal</u>	<u>$y_p(n)$ Particular Solution</u>
$\delta(n)$	0
A(step i/p)	k
AM^n	kM^n
A_n^m	$k_0 n^m + k_1 n^{m-1} + k_2 n^{m-2} + \dots + k_m$
$A^n N^m$	$A^n [k_0 n^m + k_1 n^{m-1} + \dots + k_m]$
$A \cos \omega n$ $A \sin \omega n$	$C_1 \cos \omega n + C_2 \sin \omega n$

Where A, M, N and C_1 & C_2 are constants.

1) find the forced response of the system described by the difference equation

$$y(n) + 2y(n-1) + y(n-2) = x(n) + x(n-1) \text{ for a given i/p}$$

$$x(n) = \left(\frac{1}{2}\right)^n u(n)$$

Sol given data the difference eqⁿ is

$$y(n) + 2y(n-1) + y(n-2) = x(n) + x(n-1)$$

$$y_f(n) = y_h(n) + y_p(n)$$

$$y_h(n) = [C_1 + C_2] (-1)^n$$

$$x(n) = \left(\frac{1}{2}\right)^n u(n)$$

$$y_p(n) = k \left(\frac{1}{2}\right)^n u(n)$$

$$y_f(n) = [C_1 + C_2] (-1)^n + k \left(\frac{1}{2}\right)^n u(n)$$

$$k \left(\frac{1}{2}\right)^n + 2k \left(\frac{1}{2}\right)^{n-1} + k \left(\frac{1}{2}\right)^{n-2} = \left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{n-1}$$

$$k \left(\frac{1}{2}\right)^2 + 2k \left(\frac{1}{2}\right) + k = \left(\frac{1}{2}\right)^2 + \frac{1}{2}$$

$$\begin{matrix} n-2=0 \\ n=2 \end{matrix}$$

$$k \left(\frac{1}{4}\right) + k + k = \frac{1}{4} + \frac{1}{2}$$

$$k \left(\frac{1}{4} + 2\right) = \frac{3}{4}$$

$$k = \frac{1}{3}$$

$$y_p(n) = \frac{1}{3} \left(\frac{1}{2}\right)^n u(n)$$

In order to determine the values of C_1 & C_2 by applying initial conditions.

$$y_f(n) = [C_1 + C_2 n] (-1)^n + k \left(\frac{1}{2}\right)^n u(n)$$

$$y(0) = [C_1] (1) + \frac{1}{3} (1)$$

$$y(0) = C_1 + \frac{1}{3}$$

$$y(1) = (c_1 + c_2)(-1) + \left(\frac{1}{3}\right)\left(\frac{1}{9}\right)$$

$$y(1) = -c_1 - c_2 + \frac{1}{6}$$

$$y(n) + 2y(n-1) + y(n-2) = x(n) + x(n-1)$$

$$y(0) + 2y(-1) + y(-2) = x(0) + x(-1)$$

$$y(0) + 2(0) = 1 + 0$$

$$y(0) = 1$$

$$c_1 + \frac{1}{3} = 1$$

$$c_1 = 1 - \frac{1}{3} = \frac{2}{3}$$

$$y(1) + 2y(0) + y(-1) = x(1) + x(0)$$

$$y(1) + 2(1) + 0 = \frac{1}{2} + 1$$

$$y(1) + 2 = \frac{3}{2}$$

$$y(1) = -\frac{1}{2}$$

$$-c_1 - c_2 + \frac{1}{6} = -\frac{1}{2}$$

$$-c_2 = -\frac{1}{2} - \frac{1}{6} + \frac{2}{3}$$

$$c_2 = 0$$

$$\therefore \boxed{y_f(n) = \frac{2}{3} + \frac{1}{3}\left(\frac{1}{2}\right)^n u(n)}$$

2) find the forced response of the system described by the difference equation $y(n) - 4y(n-1) + 4y(n-2) = x(n) - x(n-1)$

when the i/p is $x(n) = (-1)^n u(n)$.

sol given difference equation

$$y(n) - 4y(n-1) + 4y(n-2) = x(n) - x(n-1)$$

115

$$y_f(n) = y_h(n) + y_p(n)$$

$$y_h(n) = [C_1 + C_2 n](2)^n$$

$$y_p(n) = K(-1)^n u(n)$$

$$y_f(n) = [C_1 + C_2 n](2)^n + K(-1)^n u(n)$$

$$K(-1)^n - 4K(-1)^{n-1} + 4K(-1)^{n-2} = (-1)^n - (-1)^{n-1}$$

$$n-2=0$$

$$n=2$$

$$K(-1)^2 - 4K(-1)^1 + 4K(-1)^0 = (-1)^2 - (-1)^1$$

$$K + 4K + 4K = 1 + 1$$

$$9K = 2$$

$$K = 2/9$$

$$y_p(n) = \frac{2}{9}(-1)^n u(n)$$

In order to determine the values of C_1 & C_2 by applying initial conditions

$$y_f(n) = [C_1 + C_2 n](2)^n + K(-1)^n u(n)$$

$$y(0) = [C_1 + 0](2)^0 + \frac{2}{9}(-1)^0 \cdot 1$$

$$y(0) = C_1 + \frac{2}{9}$$

$$y(1) = (C_1 + C_2)(2) + \frac{2}{9}(-1)^1$$

$$y(1) = (C_1 + C_2)(2) - \frac{2}{9}$$

$$y(n) - 4y(n-1) + 4y(n-2) = x(n) - x(n-1)$$

$$y(0) - 4y(-1) + 4y(-2) = x(0) - x(-1)$$

$$y(0) = 1$$

$$C_1 + \frac{2}{9} = 1$$

$$C_1 = 1 - 2/9$$

$$C_1 = 7/9$$

$$y(1) - 4y(0) + 4y(-1) = x(1) - x(0)$$

$$y(1) - 4 + 0 = (-1) - 1$$

$$y(1) = -2 + 4$$

$$y(1) = 2$$

$$2C_1 + 2C_2 - 2/9 = 2$$

$$2(7/9) + 2C_2 = 2 + 2/9$$

$$\frac{14}{9} + 2C_2 = 20/9$$

$$2C_2 = \frac{20}{9} - \frac{14}{9} = 6/9$$

$$C_2 = 1/3$$

$$y(n) = \left[\frac{7}{9} + \frac{1}{3}n \right] (2)^n + \left(\frac{2}{9} \right) (-1)^n u(n)$$

Frequency Response Analysis of Discrete Time Systems:-

The output $y(n)$ of any linear time invariant system to an input signal $x(n)$ can be obtained by using Convolution sum $y(n) = \sum_{k=-\infty}^{\infty} h(k) x(n-k)$.

Where $h(n)$ impulse response of the system. Let us consider a complex exponential signal $x(n) = e^{j\omega n}$ as input to the system then the output is given by

$$y(n) = \sum_{k=-d}^d h(k) e^{j\omega(n-k)}$$

$$= \sum_{k=-d}^d h(k) e^{j\omega n} \cdot e^{-j\omega k}$$

$$y(n) = e^{j\omega n} \sum_{k=-d}^d h(k) e^{-j\omega k}$$

$$y(n) = e^{j\omega n} H(e^{j\omega})$$

$$\text{where } H(e^{j\omega}) = \sum_{k=-d}^d h(k) e^{-j\omega k}$$

The quantity $H(e^{j\omega})$ is called frequency response of the system. The frequency response is a complex valued function it can be expressed in polar form as $H(e^{j\omega}) = |H(e^{j\omega})| e^{j\theta(\omega)}$.

where $|H(e^{j\omega})|$ is called the magnitude response and it is an even function of ' ω '.

$$\text{i.e., } |H(e^{j\omega})| = |H(e^{-j\omega})|$$

* $\theta(\omega)$ is called phase response of the system and it is an odd function of ' ω '.

$$\text{i.e., } \theta(\omega) = -\theta(-\omega)$$

Discrete Fourier Series

Consider a Periodic Sequence $x_p(n)$ with a period N .

i.e. $x_p(n) = x_p(n + lN)$

where l is an Integer.

The Periodic Sequence Can be represented as weighted sum of complex exponential whose frequencies are integer multiples of fundamental frequency $2\pi/N$.

$$\therefore e^{j2\pi kn/N} = e^{j2\pi k(n+lN)/N}$$

$$\therefore x_p(n) \text{ can be defined as } \boxed{x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_p(k) e^{j2\pi kn/N}}$$

where $X_p(k)$ are discrete Fourier series Co-efficient for

$$k = 0 \text{ to } N-1 \\ = 0, 1, 2, 3, \dots, N-1$$

$$\boxed{X_p(k) = \sum_{n=0}^{N-1} x_p(n) e^{-j2\pi kn/N}}$$

$\therefore$ The above two equations are the discrete Fourier Series pair

It can be represented as $\text{DFS}[x_p(n)] = X_p(k)$

Properties:1) Linearity Property :-

$$\text{If } \text{DFS}[x_1(n)] = X_1(k)$$

$$\text{DFS}[x_2(n)] = X_2(k)$$

$$\text{Then } \text{DFS}[a \cdot x_1(n) + b \cdot x_2(n)] = aX_1(k) + bX_2(k)$$

2) Time Shifting Property

$$\text{If } \text{DFS}[x_p(n)] = X_p(k)$$

$$\text{Then } \text{DFS}[x_p(n-m)] = e^{-j2\pi km/N} X_p(k)$$

Periodic Convolution Property:-

$$\text{If Dfs } [x_{1p}(n)] = X_{1p}(k)$$

$$\text{Dfs } [x_{2p}(n)] = X_{2p}(k)$$

$$\text{Then Dfs } [x_{1p}(n) * x_{2p}(n)] = X_{1p}(k) X_{2p}(k)$$

Symmetry Property:-

$$\text{If Dfs } [x_p(n)] = X_p(k)$$

$$\text{Dfs } [x_p^*(n)] = X_p^*(-k)$$

$$\text{Dfs } [x_p^*(-n)] = X_p^*(k)$$

$$\text{Then Dfs } [\text{Re } x_p(n)] = \text{Dfs } \left[\frac{x_p(n) + x_p^*(-n)}{2} \right] = \frac{X_p(k) + X_p^*(-k)}{2} = \text{Re } X_p(k)$$

$$\text{Dfs } [\text{Im } x_p(n)] = \text{Dfs } \left[\frac{x_p(n) - x_p^*(-n)}{2} \right] = \frac{X_p(k) - X_p^*(-k)}{2} = \text{Im } X_p(k)$$

Consider: $x_p(n) = x_{pe}(n) + x_{po}(n)$

$$\text{then } x_{pe}(n) = \frac{x_p(n) + x_p^*(-n)}{2}$$

$$x_{po}(n) = \frac{x_p(n) - x_p^*(-n)}{2}$$

$$\text{Dfs } [x_{pe}(n)] = \text{Dfs } \left[\frac{x_p(n) + x_p^*(-n)}{2} \right] = \frac{X_p(k) + X_p^*(-k)}{2} = \text{Re } X_p(k)$$

$$\text{Dfs } [x_{po}(n)] = \text{Dfs } \left[\frac{x_p(n) - x_p^*(-n)}{2} \right] = \frac{X_p(k) - X_p^*(-k)}{2} = \text{Im } X_p(k)$$

Discrete Fourier Transform (DFT):-

The discrete Fourier transform of sequence $x(n)$ is given by

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} \quad \text{and}$$
$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

Let us (Consider) define a term $W_N = e^{j2\pi/N}$ called twiddle factor.

$$W_N = e^{j2\pi/N} = \cos 2\pi/N + j \sin 2\pi/N$$

The magnitude of twiddle factor will be

$$|W_N| = |\cos 2\pi/N + j \sin 2\pi/N| \text{ and}$$

$$\text{Phase } W_N = \angle e^{j2\pi/N}$$

In terms of twiddle factor the sequence of Discrete Fourier transform of $x(n)$ is defined as

$$\boxed{\begin{aligned} X(k) &= \sum_{n=0}^{N-1} x(n) W_N^{nk} \\ x(n) &= \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-nk} \end{aligned}}$$

$\therefore$ The representation of discrete Fourier Transform is

$$\text{DFT } [x(n)] = X(k)$$

$$x(n) = \text{IDFT } [X(k)]$$

1) Find the DFT of a sequence $x(n) = \{1, 1, 0, 0\}$ and IDFT of the sequence $X(k) = \{1, 0, 1, 0\}$

Sol Given sequence $x(n) = \{1, 1, 0, 0\}$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{j2\pi kn/N}$$

Given N and $L \Rightarrow N=L=4$

$$\text{Consider } X(0) = \sum_{n=0}^3 x(n) e^{j2\pi(0)n/4}$$

$$= x(0) + x(1) + x(2) + x(3)$$

$$= 1 + 1 + 0 + 0$$

$$= 2$$

$$\begin{aligned}
 x(1) &= \sum_{n=0}^3 x(n) e^{-j2\pi(1)n/4} \\
 &= \sum_{n=0}^3 x(n) e^{-j\pi n/2} \\
 &= x(0) e^{-j\pi(0)/2} + x(1) e^{-j\pi(1)/2} + x(2) e^{-j\pi(2)/2} + x(3) e^{-j\pi(3)/2} \\
 &= 1 + 1 \cdot e^{-j\pi/2} + 0 + 0 \\
 &= 1 + 1 [\cos(\pi/2) - j\sin(\pi/2)] \\
 &= 1 + [0 - j \cdot 1] \\
 &= 1 - j.
 \end{aligned}$$

$$\begin{aligned}
 x(2) &= \sum_{n=0}^3 x(n) e^{-j2\pi(2)n/4} \\
 &= \sum_{n=0}^3 x(n) e^{-j\pi n} \\
 &= x(0) e^{-j\pi(0)} + x(1) e^{-j\pi(1)} + x(2) e^{-j\pi(2)} + x(3) e^{-j\pi(3)} \\
 &= 1 + 1 \cdot e^{-j\pi} + 0 + 0 \\
 &= 1 + [\cos\pi - j\sin\pi] \\
 &= 1 + [-1 - j(0)] \\
 &= 1 - 1 = 0
 \end{aligned}$$

$$\begin{aligned}
 x(3) &= \sum_{n=0}^3 x(n) e^{-j2\pi(3)n/4} \\
 &= \sum_{n=0}^3 x(n) e^{-j3\pi n/2} \\
 &= x(0) e^{-j3\pi(0)/2} + x(1) e^{-j3\pi(1)/2} + x(2) e^{-j3\pi(2)/2} + x(3) e^{-j3\pi(3)/2} \\
 &= 1 + 1 \cdot e^{-j3\pi/2} \\
 &= 1 + 1 [\cos 3\pi/2 - j\sin 3\pi/2] \\
 &= 1 + [0 - j(-1)] \\
 &= 1 + j.
 \end{aligned}$$

$$X(k) = \{2, 1-j, 0, 1+j\}$$

* Given Sequence $y(k) = \{1, 0, 1, 0\}$

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} y(k) e^{j 2\pi k n / N}$$

$$y(0) = \frac{1}{4} \sum_{k=0}^3 y(k) e^{j 2\pi (0) n / 4}$$

$$= \frac{1}{4} \sum_{k=0}^3 y(k) \cdot 1$$

$$= \frac{1}{4} [y(0) + y(1) + y(2) + y(3)]$$

$$= \frac{1}{4} [1 + 0 + 1 + 0]$$

$$= \frac{1}{4} (2) = \frac{1}{2}$$

$$y(1) = \frac{1}{4} \sum_{k=0}^3 y(k) e^{j 2\pi (1) n / 4}$$

$$= \frac{1}{4} [y(0) e^{j 2\pi (0) / 4} + y(1) e^{j 2\pi (1) / 4} + y(2) e^{j 2\pi (2) / 4} + y(3) e^{j 2\pi (3) / 4}]$$

$$= \frac{1}{4} [1 \cdot 1 + 0 + 1 \cdot e^{j\pi} + 0]$$

$$= \frac{1}{4} [1 + \cos\pi + j\sin\pi] = \frac{1}{4} [1 + (-1 + 0)] = 0$$

$$y(2) = \frac{1}{4} \sum_{k=0}^3 y(k) e^{j 2\pi (2) n / 4}$$

$$= \frac{1}{4} \sum_{k=0}^3 y(k) e^{j\pi n}$$

$$= \frac{1}{4} [y(0) e^{j 2\pi (0) / 4} + y(1) e^{j 2\pi (1) / 4} + y(2) e^{j 2\pi (2) / 4} + y(3) e^{j 2\pi (3) / 4}]$$

$$= \frac{1}{4} [1 \cdot 1 + 0 + 1 \cdot e^{j2\pi} + 0]$$

$$= \frac{1}{4} [1 + (1 + j0)]$$

$$= \frac{1}{4} (2) = \frac{1}{2}$$

$$y(3) = \frac{1}{4} \sum_{k=0}^3 y(k) e^{j 2\pi (3) n / 4}$$

$$= \frac{1}{4} \left[\sum_{k=0}^3 y(k) e^{j 3\pi n / 2} \right]$$

$$= \frac{1}{4} [y(0) e^{j 3\pi (0) / 2} + y(1) e^{j 3\pi (1) / 2} + y(2) e^{j 3\pi (2) / 2} + y(3) e^{j 3\pi (3) / 2}]$$

$$= \frac{1}{4} [1 + 0 + 1 \cdot e^{j3\pi} + 0]$$

$$= \frac{1}{4} [1 + (\cos 3\pi + j \sin 3\pi)]$$

$$= \frac{1}{4} [1 + (-1 + j(0))]$$

$$y(3) = \frac{1}{4} [0] = 0$$

$$y(n) = \{ \frac{1}{2}, 0, \frac{1}{2}, 0 \}$$

2) Find 8 point DFT of the Sequence $x(n) = \{1, 1, 1, 1, 1, 1, 0, 0\}$
 sol given sequence is $x(n) = \{1, 1, 1, 1, 1, 1, 0, 0\}$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

$$N=8$$

$$X(0) = \sum_{n=0}^7 x(n) e^{-j2\pi (0)n/8}$$

$$N-1=8-1=7$$

$$= \sum_{n=0}^7 x(n) \cdot 1$$

$$= x(0) + x(1) + x(2) + x(3) + x(4) + x(5) + x(6) + x(7)$$

$$= 1 + 1 + 1 + 1 + 1 + 1 + 0 + 0$$

$$= 6$$

$$X(1) = \sum_{n=0}^7 x(n) e^{-j2\pi (1)n/8}$$

$$= \sum_{n=0}^7 x(n) e^{-j\pi n/4}$$

$$X(1) = x(0) e^{-j\pi(0)/4} + x(1) e^{-j\pi(1)/4} + x(2) e^{-j\pi(2)/4} + x(3) e^{-j\pi(3)/4} \\ + x(4) e^{-j\pi(4)/4} + x(5) e^{-j\pi(5)/4} + x(6) e^{-j\pi(6)/4} \\ + x(7) e^{-j\pi(7)/4}$$

$$X(1) = 1 + 1 \cdot e^{-j\pi/4} + 1 \cdot e^{-j\pi/2} + 1 \cdot e^{-j3\pi/4} + 1 \cdot e^{-j\pi} + 0 + 0 \\ + 1 \cdot e^{-j5\pi/4}$$

$$= 1 + 1 [\cos \pi/4 - j \sin \pi/4] + [\cos \pi/2 - j \sin \pi/2] + \\ [\cos 3\pi/4 - j \sin 3\pi/4] + [\cos \pi - j \sin \pi] + [\cos 5\pi/4 - j \sin 5\pi/4]$$

$$\begin{aligned}
 x(1) &= 1 + 1/\sqrt{2} - j1/\sqrt{2} + 0 - j - 1/\sqrt{2} - j/\sqrt{2} - 1 - 0 - 1/\sqrt{2} + j/\sqrt{2} \\
 &= 1 + 1/\sqrt{2} - j1/\sqrt{2} - j - 1/\sqrt{2} - j/\sqrt{2} - 1 - 1/\sqrt{2} + j/\sqrt{2} \\
 &= -1/\sqrt{2} - j/\sqrt{2} - j \\
 &= -1/\sqrt{2} - j(1 + 1/\sqrt{2})
 \end{aligned}$$

$$x(1) = -0.707 - j(1.707)$$

$$\begin{aligned}
 X(2) &= \sum_{n=0}^7 x(n) e^{-j2\pi(2)n/8} \\
 &= \sum_{n=0}^7 x(n) e^{-jn\pi/2}
 \end{aligned}$$

$$\begin{aligned}
 &= x(0) e^{-j\pi(0)/2} + x(1) e^{-j\pi/2} + x(2) e^{-j\pi(2)/2} + x(3) e^{-j\pi(3)/2} \\
 &\quad + x(4) e^{-j\pi(4)/2} + x(5) e^{-j\pi(5)/2} + x(6) e^{-j\pi(6)/2} + x(7) e^{-j\pi(7)/2}
 \end{aligned}$$

$$= 1 + e^{-j\pi/2} + 1 \cdot e^{-j\pi} + e^{-j3\pi/2} + e^{-j2\pi} + e^{-j5\pi/2} + 0 + 0$$

$$\begin{aligned}
 &= 1 + [\cos \pi/2 - j \sin \pi/2] + [\cos \pi - j \sin \pi] + [\cos 3\pi/2 - j \sin 3\pi/2] \\
 &\quad + [\cos 2\pi - j \sin 2\pi] + [\cos 5\pi/2 - j \sin 5\pi/2]
 \end{aligned}$$

$$= 1 + 0 - j - 1 - 0 + 0 + j + 1 - 0 + 0 - j$$

$$= 1 - j$$

$$X(3) = \sum_{n=0}^7 x(n) e^{-j2\pi(3)n/8}$$

$$= \sum_{n=0}^7 x(n) e^{-j3\pi(n)/4}$$

$$\begin{aligned}
 &= x(0) e^{-j3\pi(0)/4} + x(1) e^{-j3\pi/4} + x(2) e^{-j3\pi(2)/4} + x(3) e^{-j3\pi(3)/4} \\
 &\quad + x(4) e^{-j3\pi(4)/4} + x(5) e^{-j3\pi(5)/4} + x(6) e^{-j3\pi(6)/4} + x(7) e^{-j3\pi(7)/4}
 \end{aligned}$$

$$x(z) = 1 + e^{-j3\pi/4} + e^{-j3\pi/2} + e^{-j9\pi/4} + e^{-j3\pi} + e^{-j5\pi/4} + 0 + 0$$

$$= 1 + [\cos 3\pi/4 - j\sin 3\pi/4] + [\cos 3\pi/2 - j\sin 3\pi/2] +$$

$$[\cos 9\pi/4 - j\sin 9\pi/4] + [\cos 3\pi - j\sin 3\pi] +$$

$$[\cos 15\pi/4 - j\sin 15\pi/4]$$

$$x(z) = 1 + \left[\frac{1}{\sqrt{2}} - j\left(\frac{1}{\sqrt{2}}\right) \right] + [0 - j(-1)] + \left[\frac{1}{\sqrt{2}} - j\left(\frac{1}{\sqrt{2}}\right) \right]$$

$$+ (-1 - j(0)) + \left(\frac{1}{\sqrt{2}} - j\left(-\frac{1}{\sqrt{2}}\right) \right)$$

$$= 1 - \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}} + 0 + j + \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}} - 1 - 0 + \frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}} + j$$

$$= \frac{1}{\sqrt{2}} + j\left(1 - \frac{1}{\sqrt{2}}\right)$$

$$= 0.707 + 0.293j$$

$$x(4) = \sum_{n=0}^7 x(n) e^{-j2\pi(n)(4)/8}$$

$$= \sum_{n=0}^7 x(n) e^{-jn\pi}$$

$$= x(0)e^{-j(0)\pi} + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi} + x(4)e^{-j4\pi}$$

$$+ x(5)e^{-j5\pi} + x(6)e^{-j6\pi} + x(7)e^{-j7\pi}$$

$$= 1 + [\cos \pi - j\sin \pi] + [\cos 2\pi - j\sin 2\pi] + [\cos 3\pi - j\sin 3\pi]$$

$$+ [\cos 4\pi - j\sin 4\pi] + [\cos 5\pi - j\sin 5\pi] + 0 + 0$$

$$= 1 + [-1 - j(0)] + [1 - j(0)] + [-1 - j(0)] + [1 - j(0)]$$

$$+ [(1-1) - j(0)]$$

$$= 1 - 1 + 1 - 1 + 1 - 1$$

$$= 0$$

$$X(5) = \sum_{n=0}^7 x(n) e^{-j2\pi(5)n/8}$$

$$= \sum_{n=0}^7 x(n) e^{-j5\pi n/4}$$

$$= x(0)e^{-j5\pi(0)/4} + x(1)e^{-j5\pi(1)/4} + x(2)e^{-j5\pi(2)/4} + x(3)e^{-j5\pi(3)/4} + x(4)e^{-j5\pi(4)/4} + x(5)e^{-j5\pi(5)/4} + x(6)e^{-j5\pi(6)/4} + x(7)e^{-j5\pi(7)/4}$$

$$= 1 + e^{-j5\pi/4} + e^{-j5\pi/2} + e^{-j15\pi/4} + e^{-j5\pi} + e^{-j25\pi/4} + 0 + 0$$

$$= 1 + [\cos 5\pi/4 - j\sin 5\pi/4] + [\cos 5\pi/2 - j\sin 5\pi/2] + [\cos 15\pi/4 - j\sin 15\pi/4] + [\cos 5\pi - j\sin 5\pi] + [\cos 25\pi/4 - j\sin 25\pi/4]$$

$$= 1 + [-1/\sqrt{2} - j(-1/\sqrt{2})] + [0 - j] + [1/\sqrt{2} + j/\sqrt{2}] + [-1 - j(0)] + [1/\sqrt{2} - j(1/\sqrt{2})]$$

$$= 1 - 1/\sqrt{2} + j/\sqrt{2} - j + 1/\sqrt{2} + j/\sqrt{2} - 1 + 1/\sqrt{2} - j/\sqrt{2}$$

$$= j/\sqrt{2} - j + 1/\sqrt{2}$$

$$= 0.707 - j(1 - 1/\sqrt{2})$$

$$X(5) = 0.707 - j(0.293)$$

$$X(6) = \sum_{n=0}^7 x(n) e^{-j2\pi(6)n/8}$$

$$= \sum_{n=0}^7 x(n) e^{-j3\pi n/2}$$

$$= x(0)e^{-j3\pi(0)/2} + x(1)e^{-j3\pi(1)/2} + x(2)e^{-j3\pi(2)/2} +$$

$$x(3)e^{-j3\pi(3)/2} + x(4)e^{-j3\pi(4)/2} + x(5)e^{-j3\pi(5)/2} +$$

$$x(6)e^{-j3\pi(6)/2} + x(7)e^{-j3\pi(7)/2}$$

$$x(6) = 1 + e^{-j3\pi/2} + 1e^{-j3\pi} + e^{-j9\pi/2} + e^{-j6\pi} + e^{-j15\pi/2} + 0 + 0$$

$$= 1 + [\cos 3\pi/2 - j\sin 3\pi/2] + [\cos 3\pi - j\sin 3\pi] +$$

$$[\cos 9\pi/2 - j\sin 9\pi/2] + [\cos 6\pi - j\sin 6\pi] +$$

$$[\cos 15\pi/2 - j\sin 15\pi/2]$$

$$= 1 + (0 - j(-1)) + (-1 - j(0)) + (0 - j(1)) + (1 - j(0)) +$$

$$(0 - j(-1))$$

$$x(6) = 1 + j - 1 - j + 1 + j$$

$$= 1 + j$$

$$x(7) = \sum_{n=0}^7 x(n) e^{-j\pi(7)n/4}$$

$$= \sum_{n=0}^7 x(n) e^{-j7\pi n/4}$$

$$= x(0)e^{-j7\pi(0)/4} + x(1)e^{-j7\pi/4} + x(2)e^{-j7\pi(2)/4} +$$

$$x(3)e^{-j7\pi(3)/4} + x(4)e^{-j7\pi(4)/4} + x(5)e^{-j7\pi(5)/4} +$$

$$x(6)e^{-j7\pi(6)/4} + x(7)e^{-j7\pi(7)/4}$$

$$= 1 + e^{-j7\pi/4} + e^{-j7\pi/2} + e^{-j21\pi/4} + e^{-j7\pi} + e^{-j35\pi/4} + 0 + 0$$

$$= 1 + [\cos 7\pi/4 - j\sin 7\pi/4] + [\cos 7\pi/2 - j\sin 7\pi/2] +$$

$$[\cos 21\pi/4 - j\sin 21\pi/4] + [\cos 7\pi - j\sin 7\pi] +$$

$$[\cos 35\pi/4 - j\sin 35\pi/4]$$

$$= 1 + [1/\sqrt{2} - j(-1/\sqrt{2})] + [0 - j(-1)] + [-1/\sqrt{2} - j(-1/\sqrt{2})] +$$

$$(-1-j(0)) + (-1/\sqrt{2} - j(1/\sqrt{2}))$$

$$= 1 + 1/\sqrt{2} + j/\sqrt{2} + j - 1/\sqrt{2} + j/\sqrt{2} - 1 - 1/\sqrt{2} - j/\sqrt{2} - j - 1/\sqrt{2} + j/\sqrt{2}$$

$$x(7) = -0.707 + j(1.707)$$

$$x(k) = \{6, -0.707 - j1.707, -j, 0.707 + 0.293j, 0, 0.707 - 0.293j, 1+j, -0.707 + j1.707\}$$

3) Find IDFT of Sequence $x(k) = \{5, 0, 1-j, 0, 1, 0, 1+j, 0\}$
Sol given Sequence $x(k) = \{5, 0, 1-j, 0, 1, 0, 1+j, 0\}$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} x(k) e^{j2\pi kn/N}$$

$$N=8$$

$$N-1 = 8-1 = 7$$

$$x(0) = \frac{1}{8} \sum_{k=0}^7 x(k) e^{j2\pi(0)k/8}$$

$$= \frac{1}{8} \sum_{k=0}^7 x(k) \cdot 1$$

$$x(0) = \frac{1}{8} [x(0) + x(1) + x(2) + x(3) + x(4) + x(5) + x(6) + x(7)]$$

$$= \frac{1}{8} [5 + 0 + 1-j + 0 + 1 + 0 + 1+j + 0]$$

$$= \frac{1}{8} [8]$$

$$= 1$$

$$x(1) = \frac{1}{8} \sum_{k=0}^7 x(k) e^{j2\pi k(1)/8}$$

$$= \frac{1}{8} \sum_{k=0}^7 x(k) e^{j\pi k/4}$$

$$= \frac{1}{8} [x(0) e^{j\pi(0)/4} + x(1) e^{j\pi(1)/4} + x(2) e^{j\pi(2)/4} + x(3) e^{j\pi(3)/4} + x(4) e^{j\pi(4)/4} + x(5) e^{j\pi(5)/4} + x(6) e^{j\pi(6)/4} + x(7) e^{j\pi(7)/4}]$$

$$x(1) = \frac{1}{8} [5 + 0 + (1-j)e^{j\pi/2} + 0e^{j3\pi/4}e^{j\pi} + 0e^{j5\pi/4} + (1+j)e^{j3\pi/2} + 0e^{j7\pi/4}]$$

$$= \frac{1}{8} [5 + (1-j)[\cos \pi/2 + j\sin \pi/2] + [\cos \pi + j\sin \pi] + (1+j)[\cos 3\pi/2 + j\sin 3\pi/2]]$$

$$= \frac{1}{8} [5 + (1-j)(0+j) + (-1+j(0)) + (1+j)(0+j(-1))] \quad \checkmark$$

$$= \frac{1}{8} [5 + j - j^2 - 1 + (1+j)(-j)]$$

$$= \frac{1}{8} [5 + j - j^2 - 1 - j^2 - j]$$

$$= \frac{1}{8} [5 - 1 + 1 + 1]$$

$$= 6/8 = 3/4 = 0.75$$

$$x(2) = \frac{1}{8} \sum_{k=0}^7 x(k) e^{-j2\pi k(2)/82}$$

$$= \frac{1}{8} \sum_{k=0}^7 x(k) e^{j\pi k/2}$$

$$x(2) = \frac{1}{8} [x(0)e^{j\pi(0)/2} + x(1)e^{j\pi(1)/2} + x(2)e^{j\pi(2)/2} + x(3)e^{j\pi(3)/2} + x(4)e^{j\pi(4)/2} + x(5)e^{j5\pi/2} + x(6)e^{j\pi(6)/2} + x(7)e^{j7\pi/2}]$$

$$= \frac{1}{8} [5 + 0 + (1-j)e^{j\pi} + 0 + e^{j2\pi} + 0 + (1+j)e^{j3\pi} + 0]$$

$$= \frac{1}{8} [5 + (1-j)[\cos \pi + j\sin \pi] + (\cos 2\pi + j\sin 2\pi) + (1+j)(\cos 3\pi + j\sin 3\pi)]$$

$$= \frac{1}{8} [5 + (1-j)(-1+j(0)) + (1+j(0)) + (1+j)(-1+j(0))] \quad \checkmark$$

$$= \frac{1}{8} [5 - 1 + j + 1 + (1+j)(1)]$$

$$= \frac{1}{8} [4 + j + 1 - j]$$

$$= \frac{4}{8} = \frac{1}{2} = 0.5$$

$$X(3) = \frac{1}{8} \sum_{k=0}^7 x(k) e^{j2\pi k(3)/8}$$

$$= \frac{1}{8} \sum_{k=0}^7 x(k) e^{j3\pi k/4}$$

$$= \frac{1}{8} [x(0) e^{j3\pi(0)/4} + x(1) e^{j3\pi(1)/4} + x(2) e^{j3\pi(2)/4} + x(3) e^{j3\pi(3)/4} \\ + x(4) e^{j3\pi(4)/4} + x(5) e^{j3\pi(5)/4} + x(6) e^{j3\pi(6)/4} + x(7) e^{j3\pi(7)/4}]$$

$$X(3) = \frac{1}{8} [5 + 0 + (1-j) e^{j3\pi/2} + 0 + e^{j3\pi} + 0 + (1+j) e^{j9\pi/2} + 0]$$

$$= \frac{1}{8} [5 + (1-j) [\cos 3\pi/2 + j \sin 3\pi/2] + (\cos 3\pi + j \sin 3\pi) \\ + (1+j) (\cos 9\pi/2 + j \sin 9\pi/2)]$$

$$= \frac{1}{8} [5 + (1-j)(0 + j(-1)) + (-1) + j(0) + (1+j)(0 + j)]$$

$$= \frac{1}{8} [5 + (1-j)(-j) + (-1) + (1+j)(j)]$$

$$= \frac{1}{8} [5 - j + j^2 - 1 + j + j^2]$$

$$= \frac{1}{8} [4 - 1 - 1] = \frac{1}{8}(2) = \frac{1}{4} = 0.25$$

$$X(4) = \frac{1}{8} \sum_{k=0}^7 x(k) e^{j2\pi k(4)/8}$$

$$= \frac{1}{8} \sum_{k=0}^7 x(k) e^{j\pi k}$$

$$= \frac{1}{8} [x(0) e^{j\pi(0)} + x(1) e^{j\pi(1)} + x(2) e^{j\pi(2)} + x(3) e^{j\pi(3)} +$$

$$x(4)e^{j4\pi} + x(5)e^{j5\pi} + x(6)e^{j6\pi} + x(7)e^{j7\pi}]$$

$$x(4) = \frac{1}{8} [5 + 0 + (1-j)(\cos 2\pi + j\sin 2\pi) + 0 + (\cos 4\pi + j\sin 4\pi) + (1+j)(\cos 6\pi + j\sin 6\pi)]$$

$$= \frac{1}{8} [5 + (1+j)(1+j(0)) + (1+j(0)) + (1+j)(1+j(0))]$$

$$= \frac{1}{8} [5 + 1 - j + j + 1 + j]$$

$$= 8(1/8) = 1$$

$$x(5) = \frac{1}{8} \sum_{k=0}^7 x(k) e^{j7\pi k(5)/84}$$

$$= \frac{1}{8} \sum_{k=0}^7 x(k) e^{j5\pi k/4}$$

$$x(5) = \frac{1}{8} [x(0)e^{j5\pi(0)/4} + x(1)e^{j5\pi(1)/4} + x(2)e^{j5\pi(2)/4} + x(3)e^{j5\pi(3)/4} + x(4)e^{j5\pi(4)/4} + x(5)e^{j5\pi(5)/4} + x(6)e^{j5\pi(6)/4} + x(7)e^{j5\pi(7)/4}]$$

$$= \frac{1}{8} [5 + 0 + (1-j)e^{j5\pi/2} + 0 + e^{j5\pi} + 0 + (1+j)e^{j15\pi/2} + 0]$$

$$= \frac{1}{8} [5 + (1-j)[\cos 5\pi/2 + j\sin 5\pi/2] + (\cos 5\pi + j\sin 5\pi) + (1+j)(\cos 15\pi/2 + j\sin 15\pi/2)]$$

$$= \frac{1}{8} [5 + (1-j)(0 + j(1)) + (-1 + j(0)) + (1+j)(0 + j(-1))]$$

$$= \frac{1}{8} [5 + j - j^2 - 1 - j - j^2]$$

$$= \frac{1}{8} [4 - 2j^2] = \frac{1}{8} [4 + 2] = 6/8$$

$$x(5) = 0.75$$

$$X(6) = \frac{1}{8} \sum_{k=0}^7 X(k) e^{j2\pi k(6)/8} =$$

$$= \frac{1}{8} \sum_{k=0}^7 X(k) e^{j3\pi k/2}$$

$$= \frac{1}{8} \left[X(0) e^{j3\pi(0)/2} + X(1) e^{j3\pi/2} + X(2) e^{j3\pi(2)/2} + X(3) e^{j3\pi(3)/2} \right. \\ \left. + X(4) e^{j3\pi(4)/2} + X(5) e^{j3\pi(5)/2} + X(6) e^{j3\pi(6)/2} + X(7) e^{j3\pi(7)/2} \right]$$

$$= \frac{1}{8} [5 + 0 + (1-j) e^{j3\pi} + 0 + e^{j6\pi} + 0 + (1+j) e^{j9\pi} + 0]$$

$$= \frac{1}{8} [5 + (1-j) [\cos 3\pi + j \sin 3\pi] + [\cos 6\pi + j \sin 6\pi] + (1+j) [\cos 9\pi + j \sin 9\pi]]$$

$$= \frac{1}{8} [5 + (1-j) (-1 + j(0)) + (1 + j(0)) + (1+j) (-1 + j(0))]$$

$$= \frac{1}{8} [5 - 1 + j + 1 - j - 1]$$

$$= 4/8 = 1/2 = 0.5$$

$$X(7) = \frac{1}{8} \sum_{k=0}^7 X(k) e^{j2\pi k(7)/8}$$

$$= \frac{1}{8} \sum_{k=0}^7 X(k) e^{j7\pi k/4}$$

$$= \frac{1}{8} \left[X(0) e^{j7\pi(0)/4} + X(1) e^{j7\pi(1)/4} + X(2) e^{j7\pi(2)/4} + X(3) e^{j7\pi(3)/4} \right. \\ \left. + X(4) e^{j7\pi(4)/4} + X(5) e^{j7\pi(5)/4} + X(6) e^{j7\pi(6)/4} + X(7) e^{j7\pi(7)/4} \right]$$

$$= \frac{1}{8} [5 + 0 + (1-j) e^{j7\pi/2} + 0 + e^{j7\pi} + (1+j) e^{j21\pi/2} + 0]$$

$$= \frac{1}{8} \left[5 + (1-j) [\cos 7\pi/2 + j \sin 7\pi/2] + [\cos 7\pi + j \sin 7\pi] + (1+j) [\cos 21\pi/2 + j \sin 21\pi/2] \right]$$

$$= \frac{1}{8} \left[5 + (1-j)(0+j(-1)) + (-1+j(0)) + (1+j)(0+j(1)) \right]$$

$$= \frac{1}{8} \left[5 + (1-j)(-j) + (-1) + (1+j)(j) \right]$$

$$= \frac{1}{8} \left[5 - j + j^2 - 1 + j + j^2 \right]$$

$$= \frac{1}{8} \left[4 + 2j^2 \right] = \frac{1}{8} [4 - 2] = \frac{1}{8} [2] = \frac{1}{4} = 0.25$$

$$x(n) = \{1, 0.75, 0.5, 0.25, 1, 0.75, 0.5, 0.25\}$$

Zero Padding:-

Consider a Sequence $x(n)$ having length 'L' is given by $x(n) = \{x(0), x(1), x(2), \dots, x(L-1)\}$ to improve the frequency Resolution of discrete Fourier transform $(2\pi/N)$. Zero padding is require that means adding no. of zero's to the given Sequence $x(n)$.

1) find the DFT of a Sequence $x(n) = \{1 \text{ for } 0 \leq n \leq 2, 0 \text{ otherwise}\}$ for $N=4$ & $N=8$

Sol for $N=4$

$$x(n) = \{1, 1, 1, 0\}$$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

$$X(0) = \sum_{n=0}^3 x(n) e^{-j2\pi(0)n/4}$$

$$X(0) = \sum_{n=0}^3 x(n) \cdot 1$$

$$= x(0) + x(1) + x(2) + x(3)$$

$$= 1 + 1 + 1 + 0$$

$$X(0) = 3$$

$$X(1) = \sum_{n=0}^3 x(n) e^{-j2\pi(1)n/4}$$

$$= \sum_{n=0}^3 x(n) e^{-j\pi n/2}$$

$$= x(0)e^{-j\pi(0)/2} + x(1)e^{-j\pi/2} + x(2)e^{-j\pi(2)/2} + x(3)e^{-j\pi(3)/2}$$

$$= 1 + e^{-j\pi/2} + e^{-j\pi} + 0$$

$$= 1 + [\cos \pi/2 - j \sin \pi/2] + [\cos \pi - j \sin \pi]$$

$$= 1 + [0 - j] + [-1 - j(0)]$$

$$= 1 - j - 1$$

$$= -j$$

$$X(2) = \sum_{n=0}^3 x(n) e^{-j2\pi(2)n/4}$$

$$= \sum_{n=0}^3 x(n) e^{-j\pi n}$$

$$= x(0)e^{-j\pi(0)} + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi}$$

$$= 1 + 1[\cos \pi - j \sin \pi] + [\cos 2\pi - j \sin 2\pi] + 0$$

$$= 1 + [-1 - j(0)] + [1 - j(0)]$$

$$= 1 - 1 + 1 = 1$$

$$X(3) = \sum_{n=0}^3 x(n) e^{-j2\pi(3)n/4}$$

$$= \sum_{n=0}^3 x(n) e^{-j3\pi n/2}$$

$$= x(0)e^{-j3\pi(0)/2} + x(1)e^{j3\pi(1)/2} + x(2)e^{-j3\pi(2)/2} + x(3)e^{-j3\pi(3)/2}$$

$$= 1 + [\cos 3\pi/2 - j\sin 3\pi/2] + [\cos 3\pi - j\sin 3\pi] + 0$$

$$= 1 + [0 - j(-1)] + [-1 - j(0)]$$

$$= 1 + j - 1$$

$$= j$$

$$x(k) = \{3, -j, 1, j\}$$

(ii) for $N=8$

$$x(n) = \{1, 1, 1, 1, 0, 0, 0, 0\}$$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{j2\pi kn/8}$$

Circular Convolution:-

The linear Convolution of two Sequences $x(n)$ of 'L' no of Samples and $h(n)$ of 'M' no of Samples produce a result $y(n)$ which contains $N = L + M - 1$ incase of Circular Convolution if $x(n)$ contains 'L' no. of Samples and $h(n)$ has 'M' no of Samples then we can perform Circular Convolution b/w the two Sequence using

$$N = \max(L, M)$$

To find the Circular Convolution of two Sequence their are two methods are use

- 1) Concentric Circle method
- 2) Matrix multiplication method

(i) Concentric Circle method:-

Given two Sequences $x_1(n)$ and $x_2(n)$ the Circular Convolution of these two Sequence $x_3(n)$ $x_3(n) = x_1(n) \otimes x_2(n)$ can be found by using the following steps.

1. Graph 'n' Samples of $x_1(n)$ as equally spaced points around an outer circle in counter clock-wise direction.
2. Start at the same point as $x_1(n)$ graph 'n' Samples of $x_2(n)$ as equally spaced points around as inner circle in clockwise direction.
3. Multiply the corresponding Samples on the two Circles and sum the products to produce an Outputs.

4. Rotate the inner circle one sample at a time in count clockwise direction and go to step 3 to obtain the next value of output.

5. Repeat step 4 until the inner circle first sample lines up with the first sample of the exterior circle once again.

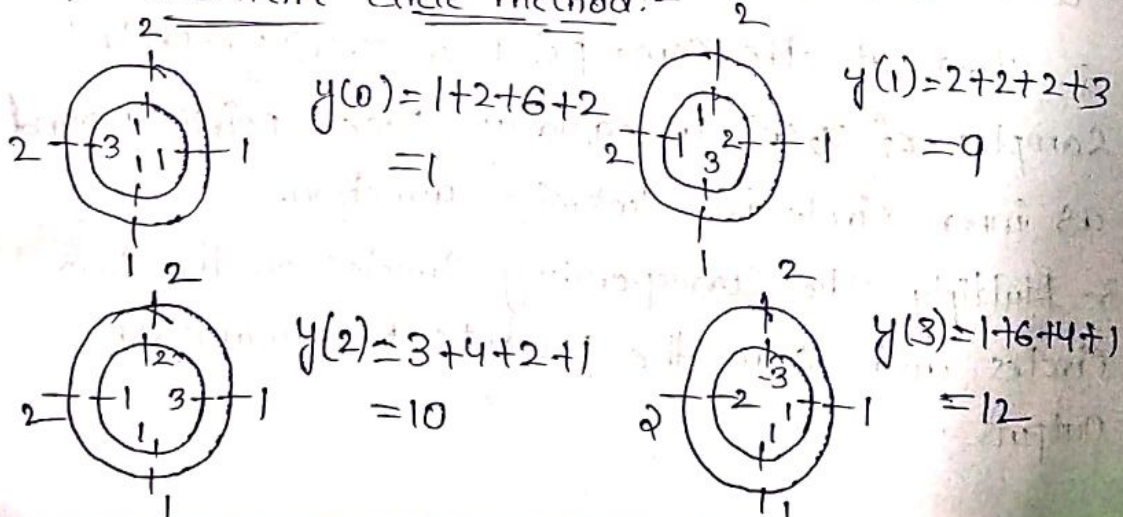
(2) Matrix Multiplication Method:-

In this method, the Circular Convolution of two sequence $x_1(n)$ & $x_2(n)$ can be obtained by representing the sequence in matrix form that is the sequence $x_2(n)$ is repeated via circular shift of samples and represented in $N \times N$ matrix form the sequence $x_1(n)$ is represented as column matrix. the multiplication of these two matrices gives the sequences $x_3(n)$.

1) find Circular Convolution of two sequence $x_1(n) = \{1, 2, 2, 1\}$, $x_2(n) = \{1, 2, 3, 1\}$ using concentric circle method, matrix multiplication method.

A) Given $x_1(n) = \{1, 2, 2, 1\}$
 $x_2(n) = \{1, 2, 3, 1\}$

1) Concentric circle method:-



$$y(n) = \{11, 9, 10, 12\}$$

2) Matrix Multiplication method

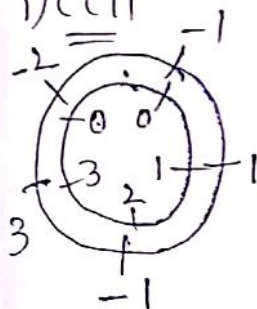
$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+2+6+2 \\ 2+2+3+1 \\ 2+4+3+1 \\ 1+4+6+1 \end{bmatrix} = \begin{bmatrix} 11 \\ 9 \\ 10 \\ 12 \end{bmatrix}$$

2) find the Circular Convolution of two finite duration of Sequence $x_1(n) = \{1, -1, -2, 3, -1\}$ and $x_2(n) = \{1, 2, 3\}$

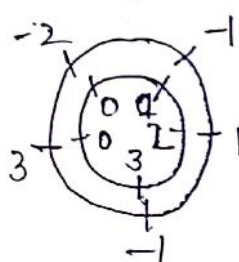
a) given $x_1(n) = \{1, -1, -2, 3, -1\}$

$$x_2(n) = \{1, 2, 3, 0, 0\}$$

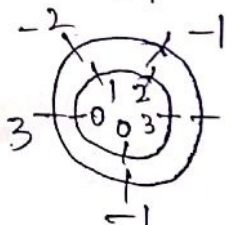
1) CCM



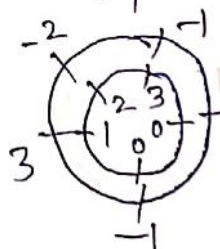
$$y(0) = 1 + 0 + 9 + 2 + 0 = 8$$



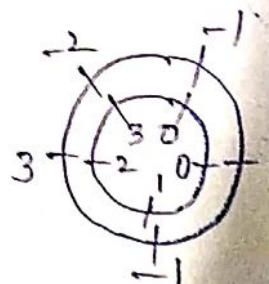
$$y(1) = 2 - 1 + 0 + 0 - 3 = -2$$



$$y(2) = 3 - 2 - 2 + 0 + 0 = -1$$



$$y(3) = 0 - 3 - 4 + 3 + 0 = -4$$



$$y(4) = 0 + 0 - 6 + 6 - 1 = -1$$

$$y(n) = \{8, -2, -1, -4, -1\}$$

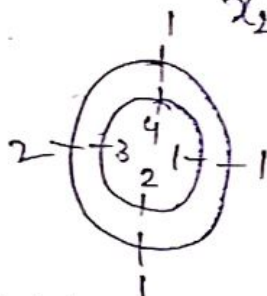
2) MMH

$$\begin{bmatrix} 1 & -1 & 3 & -2 & -1 \\ -1 & 1 & -1 & 3 & -2 \\ -2 & -1 & 1 & -1 & 3 \\ 3 & -2 & -1 & 1 & -1 \\ -1 & 3 & -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1-2+9+0+0 \\ -1+2-3+0+0 \\ -2-2+3+0+0 \\ 3-4-3+0+0 \\ -1+6-6+0+0 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ -1 \\ -4 \\ -1 \end{bmatrix}$$

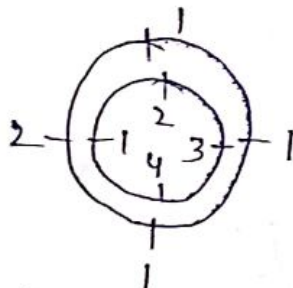
$$y(n) = \{8, -2, -1, -4, -1\}$$

3) perform the circular convolution of the following sequence (i) $x_1(n) = \{1, 1, 2, 1\}$, $x_2(n) = \{1, 2, 3, 4\}$, (ii) $x_1(n) = \{1, 2, 3, 1\}$ and $x_2(n) = \{4, 3, 2, 2\}$

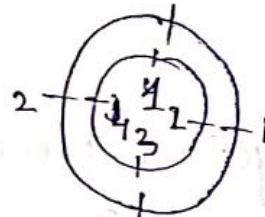
Sol (i) $x_1(n) = \{1, 1, 2, 1\}$, $x_2(n) = \{1, 2, 3, 4\}$



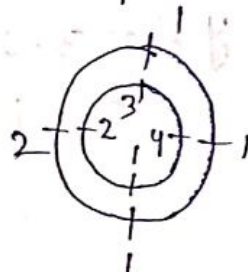
$$y(0) = 1 + 4 + 6 + 2 = 13$$



$$y(2) = 3 + 2 + 2 + 4 = 11$$



$$y(1) = 2 + 1 + 8 + 3 = 14$$



$$y(3) = 4 + 3 + 4 + 1 = 12$$

$$y(n) = \{13, 14, 11, 12\}$$

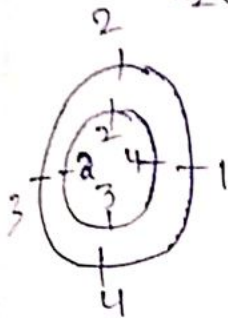
2) MM

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1+2+6+4 \\ 1+2+3+8 \\ 2+2+3+4 \\ 1+4+3+4 \end{bmatrix} = \begin{bmatrix} 13 \\ 14 \\ 11 \\ 12 \end{bmatrix}$$

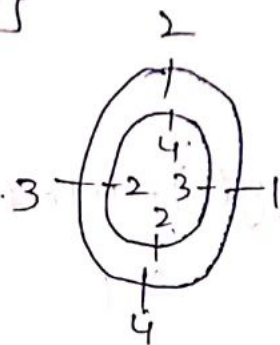
$$y(n) = \{13, 14, 11, 12\}$$

(ii) $x_1(n) = \{1, 2, 3, 1\}$

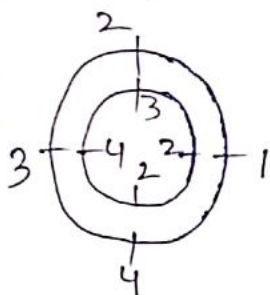
$$x_2(n) = \{4, 3, 2, 2\}$$



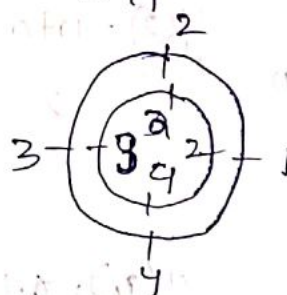
$$y(0) = 4 + 4 + 6 + 3 = 17$$



$$y(1) = 3 + 8 + 6 + 8 = 19$$



$$y(2) = 2 + 6 + 12 + 2 = 22$$



$$y(3) = 2 + 4 + 9 + 4 = 19$$

$$y(n) = \{17, 19, 22, 19\}$$

(iii)
$$\begin{bmatrix} 1 & 1 & 3 & 2 \\ 2 & 1 & 1 & 3 \\ 3 & 2 & 1 & 1 \\ 1 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4+3+6+4 \\ 8+3+2+6 \\ 12+6+2+2 \\ 4+9+4+2 \end{bmatrix} = \begin{bmatrix} 17 \\ 19 \\ 22 \\ 19 \end{bmatrix}$$

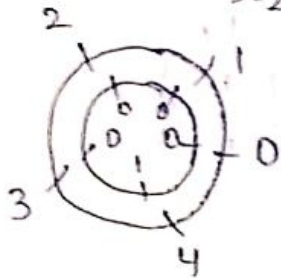
$$y(n) = \{17, 19, 22, 19\}$$

4) Consider the Sequence $x_1(n) = \{0, 1, 2, 3, 4\}$,

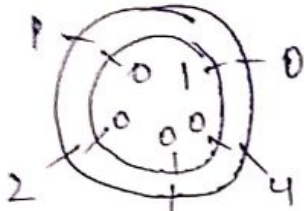
$$x_2(n) = \{0, 1, 0, 0, 0\}$$

A) given $x_1(n) = \{0, 1, 2, 3, 4\}$

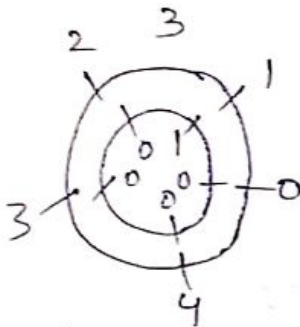
$x_2(n) = \{0, 1, 0, 0, 0\}$



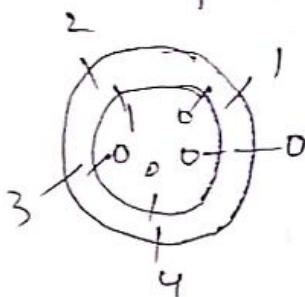
$$y(0) = 0 + 0 + 0 + 0 + 4 = 4$$



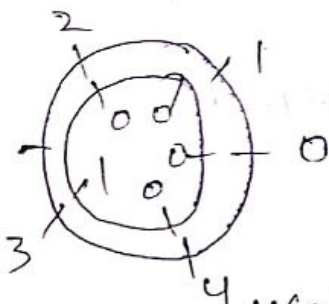
$$y(1) = 0 + 0 + 0 + 0 + 0 = 0$$



$$y(2) = 0 + 0 + 1 + 0 + 0 = 1$$



$$y(3) = 0 + 0 + 2 + 0 + 0 = 2$$



$$y(4) = 0 + 0 + 0 + 3 + 0 = 3$$

$y(n) = \{4, 0, 1, 2, 3\}$

(2) MMH

$$\begin{bmatrix} 0 & 4 & 3 & 2 & 1 \\ 1 & 0 & 4 & 3 & 2 \\ 2 & 1 & 0 & 4 & 3 \\ 3 & 2 & 1 & 0 & 4 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 + 4 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 + 0 \\ 0 + 1 + 0 + 0 + 0 \\ 0 + 0 + 2 + 0 + 0 \\ 0 + 0 + 0 + 3 + 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

Properties of DFT :-

1) Linearity Property :-

$$\text{If DFT } [x(n)] = X(k)$$

$$\text{If DFT } [x_1(n)] = X_1(k)$$

$$\text{DFT } [x_2(n)] = X_2(k) \quad \text{Then}$$

$$\text{DFT } [ax_1(n) + bx_2(n)] = aX_1(k) + bX_2(k)$$

Proof:-

$$\begin{aligned} \text{DFT } [ax_1(n) + bx_2(n)] &= \sum_{n=0}^{N-1} [ax_1(n) + bx_2(n)] e^{-j2\pi kn/N} \\ &= \sum_{n=0}^{N-1} ax_1(n) e^{-j2\pi kn/N} + \sum_{n=0}^{N-1} bx_2(n) e^{-j2\pi kn/N} \\ &= a \sum_{n=0}^{N-1} x_1(n) e^{-j2\pi kn/N} + b \sum_{n=0}^{N-1} x_2(n) e^{-j2\pi kn/N} \\ &= aX_1(k) + bX_2(k) \end{aligned}$$

$$\therefore \text{DFT } [ax_1(n) + bx_2(n)] = aX_1(k) + bX_2(k)$$

2) Time Shifting Property:

$$\text{If DFT } [x(n)] = X(k)$$

$$\text{Then DFT } [x(n-m)] = e^{-j2\pi km/N} \cdot X(k)$$

Proof:-

$$\begin{aligned} \text{DFT } [x(n-m)] &= \sum_{n=0}^{N-1} x(n-m) e^{-j2\pi kn/N} \quad \text{Let } n-m=l \\ &= \sum_{l=0}^{N-1} x(l) e^{-j2\pi k(l+m)/N} \\ &= \sum_{l=0}^{N-1} x(l) e^{-j2\pi kl/N} \cdot e^{-j2\pi km/N} \\ &= e^{-j2\pi km/N} \sum_{l=0}^{N-1} x(l) e^{-j2\pi kl/N} \\ &= e^{-j2\pi km/N} \cdot X(k) \end{aligned}$$

$$\therefore \text{DFT } [x(n-m)] = e^{-j2\pi km/N} \cdot X(k)$$

Regular Convolution:-

$$\text{If DFT}[x_1(n)] = X_1(k)$$

$$\text{DFT}[x_2(n)] = X_2(k)$$

$$\text{then DFT}[x_1(n) \otimes x_2(n)] = X_1(k) X_2(k)$$

for

Correlation & Two Sequences:-

$$\text{If DFT}[x_1(n)] = X_1(k)$$

$$\text{DFT}[x_2(n)] = X_2(k)$$

$$\text{then DFT}[x_1(n) \cdot x_2(n)] = \frac{1}{N} [X_1(k) \otimes X_2(k)]$$

Parseval's Theorem:-

$$\text{If DFT}[x(n)] = X(k)$$

$$\text{DFT}[y(n)] = Y(k)$$

$$\text{then } \sum_{n=0}^{N-1} x(n) y^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) Y^*(k)$$

Time Reversal Property:-

$$\text{If DFT}[x(n)] = X(k)$$

$$\text{then DFT}[x(-n)] = X((-k))_N = X(N-k)$$

$$x((-n))_N = x(N-n)$$

Proof:-

$$\text{DFT}[x(n)] = \sum_{k=0}^{N-1} x(n) e^{j2\pi kn/N}$$

$$\text{let } -n =$$

$$\text{DFT}[x((-n))_N] = \text{DFT}[x(N-n)]$$

$$= \sum_{n=0}^{N-1} x(N-n) e^{j2\pi kn/N}$$

$$\text{let } N-n = m$$

$$n = N-m$$

$$= \sum_{m=0}^{N-1} x(m) e^{j2\pi k(N-m)/N}$$

$$= \sum_{m=0}^{N-1} x(m) e^{j2\pi kN/N} e^{-j2\pi km/N}$$

$$= \sum_{m=0}^{N-1} x(m) e^{-j2\pi km/N}$$

$$e^{j2\pi k} = 1, k=0, 1, 2, \dots$$

$$= \sum_{m=0}^{N-1} x(m) e^{j2\pi km/N}$$

$$= X((-k))_N = X(N-k)$$

Fast Fourier Transforms!

The DFT of a sequence can

Direct evaluation of DFT: The DFT of a sequence can be evaluated using the formula $X(k) = \sum_{n=0}^{N-1} x(n) e^{j2\pi kn/N}$

Substitution $W_N = e^{j2\pi/N}$

$$\therefore X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$= \sum_{n=0}^{N-1} [\operatorname{Re} x(n) + j \operatorname{Im} x(n)] [\operatorname{Re} W_N^{kn} + j \operatorname{Im} W_N^{kn}] \rightarrow (1)$$

$$\Rightarrow \sum_{n=0}^{N-1} \operatorname{Re} x(n) \operatorname{Re} W_N^{kn} - \sum_{n=0}^{N-1} \operatorname{Im} x(n) \operatorname{Im} W_N^{kn} + j \left[\sum_{n=0}^{N-1} \operatorname{Re} x(n) \operatorname{Im} W_N^{kn} + \sum_{n=0}^{N-1} \operatorname{Im} x(n) \operatorname{Re} W_N^{kn} \right]$$

* From eq (1) we can see that to evaluate one value of $x(k)$ the no. of complex multiplication required is N . $\therefore$ to evaluate all N values of $x(k)$ the no. of complex multiplication is $N \times N = N^2$.

* In the same way to evaluate one value of $x(k)$ the no. of complex additions required is $N-1$. To evaluate all N values of $x(k)$ the no. of complex additions required is $N(N-1)$.

* From eq (2) to evaluate all values of $x(k)$ the required no. of complex multiplications is $4N^2$.

* Similarly to evaluate $x(k)$ for all values of k then required no. of complex addition required is $4N(N-1)$.

* The direct evaluation of DFT is basically inefficient because it doesn't use symmetry and periodicity properties of twiddle factor W_N .

$$\text{Symmetry property} \rightarrow W_N^{k+N/2} = -W_N^k$$

$$\text{Periodicity property} \rightarrow W_N^{k+N} = W_N^k$$

Fast Fourier Transform

The FFT algorithm can be used the disadvantages of DFT can be overcome i.e.

Two basic properties of ripple factor.

Reduce the no. of complex multiplications & addition.

FFT algorithms are based on the fundamental principle of decomposing the computation of (discrete) DFT of a sequence of a length 'N' into successively smaller DFT.

There are basically two FFT algorithms they are.

1. Decimation in Time

2. Decimation in Frequency.

→ In Decimation in Time the FFT algorithm the time samples of the DFT are decomposed into smaller and smaller sub sequences.

→ In Decimation in Frequency approach the frequency samples of the DFT are decomposed into smaller and smaller sub sequences.

1) Decimation in Time Algorithm

This algorithm is also known as Radix-2-D

which means the no. of DFT points 'N' can be expressed as a 2^r i.e.

Let $x(n)$ be an N point sequence where 'N' is assumed to be a power of '2' decimate the sequence into two sequences of length $N/2$ where one sequence consisting of the even index value of x^n and the other of odd indexed value of

The DFT of a sequence $x(n)$ is given by $X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} x(n) W_N^{kn} \\ &= \sum_{n=0}^{N/2-1} x(2n) W_N^{k(2n)} + \sum_{n=0}^{N/2-1} x(2n+1) W_N^{k(2n+1)} \\ &= \sum_{n=0}^{N/2-1} x(2n) W_{N/2}^{kn} + \sum_{n=0}^{N/2-1} x(2n+1) W_N^{k} W_{N/2}^{kn} \\ &= \sum_{n=0}^{N/2-1} x(2n) W_{N/2}^{kn} + W_N^k \sum_{n=0}^{N/2-1} x(2n+1) W_{N/2}^{kn} \end{aligned}$$

The above equation the first sum $N/2$ DFT of the even indexed sequence and the second sum being the $N/2$ DFT of the odd indexed sequence. And $X_e(k)$ and $X_o(k)$ are periodic in k with period $N/2$.

for $k \geq N/2$ by using symmetry property

$$X(k) = X_e(k - N/2) - W_N^k X_o(k - N/2)$$

For $N=8$ The values of $X_e(k)$ and $X_o(k)$ are 4 point DFT's of even indexed sequence $x_e(n)$ and odd indexed sequence $x_o(n)$ respectively where

$$x_e(0) = x(0)$$

$$x_o(0) = x(1)$$

$$x_e(1) = x(2)$$

$$x_o(1) = x(3)$$

$$x_e(2) = x(4)$$

$$x_o(2) = x(5)$$

$$x_e(3) = x(6)$$

$$x_o(3) = x(7)$$

By substituting different values of k i.e. $k=0$ to 7

$$X(0) = X_e(0) + W_8^0 X_o(0)$$

$$X(4) = X_e(0) - W_8^0 X_o(0)$$

$$X(1) = X_e(1) + W_8^1 X_o(1)$$

$$X(5) = X_e(1) - W_8^1 X_o(1)$$

$$X(2) = X_e(2) + W_8^2 X_o(2)$$

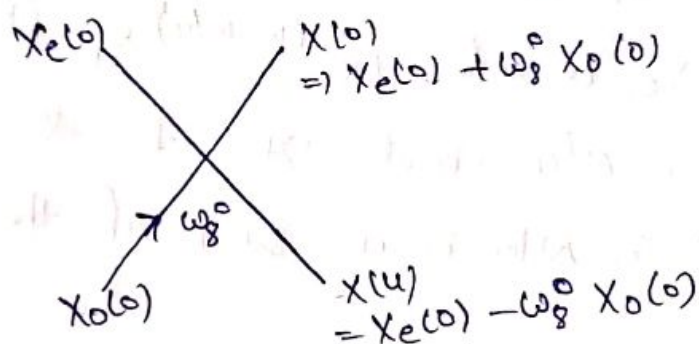
$$X(6) = X_e(2) - W_8^2 X_o(2)$$

$$X(3) = X_e(3) + W_8^3 X_o(3)$$

$$X(7) = X_e(3) - W_8^3 X_o(3)$$

$$X(4) = X_e(0) - W_8^0 X_o(0)$$

from the above set of equations we can find that $X(0)$ and $X(4)$, $X(1)$ and $X(5)$, $X(2)$ and $X(6)$, $X(3)$ and $X(7)$ have same inputs. $X(1)$ is obtained by multiplying $X_0(0)$ with ω_8^0 and adding the product to $X_2(0)$. Similarly $X(4)$ is obtained by multiplying $X_0(0)$ with ω_8^0 and subtracting the product from $X_2(0)$. This operation can be represented by a "butterfly" diagram as shown in fig below.



* The 'N/2' point DFT X_e can be expressed as a combination of 'N/4' point DFT's that $X_e(k) = X_{ee}(k) + W_N^{2k} X_{eo}(k)$
 $X_{ee}(k - N/4) = W_N^{2(k - N/4)} X_{eo}(k - N/4)$ where $X_{ee}(k)$ & 'N/4' point DFT of the even numbers of $x_e(n)$ and $X_{eo}(k)$ & 'N/4' point DFT of the odd numbers of $x_e(n)$

$$\text{Similarly } X_o(k) = X_{oe}(k) + W_N^{2k} X_{oo}(k) \\ = X_{oe}(k - N/4) - W_N^{2(k - N/4)} X_{oo}(k - N/4)$$

where $X_{oe}(k)$ & 'N/4' point DFT of the even numbers of $x_o(n)$ and $X_{oo}(k)$ & 'N/4' point DFT of the odd numbers of $x_o(n)$

* For $N=8$ the sequence $x_e(n)$ can be divided into even and odd indexed sequences.

$$x_{ee}(0) = x_e(0)$$

$$x_{ee}(1) = x_e(2)$$

$$x_{eo}(0) = x_e(1)$$

$$x_{eo}(1) = x_e(3)$$

$$X_e(0) = X_{ee}(0) + W_8^0 X_{eo}(0)$$

$$X_e(1) = X_{ee}(1) + W_8^2 X_{eo}(1)$$

$$X_e(2) = X_{ee}(0) - W_8^0 X_{eo}(0)$$

$$X_e(3) = X_{ee}(1) - W_8^2 X_{eo}(1)$$

* Similarly the odd sequence $x_o(n)$ is divided into even and odd indexed values.

$$x_{oe}(0) = x_o(0)$$

$$x_{oe}(1) = x_o(2)$$

$$x_{oo}(0) = x_o(1)$$

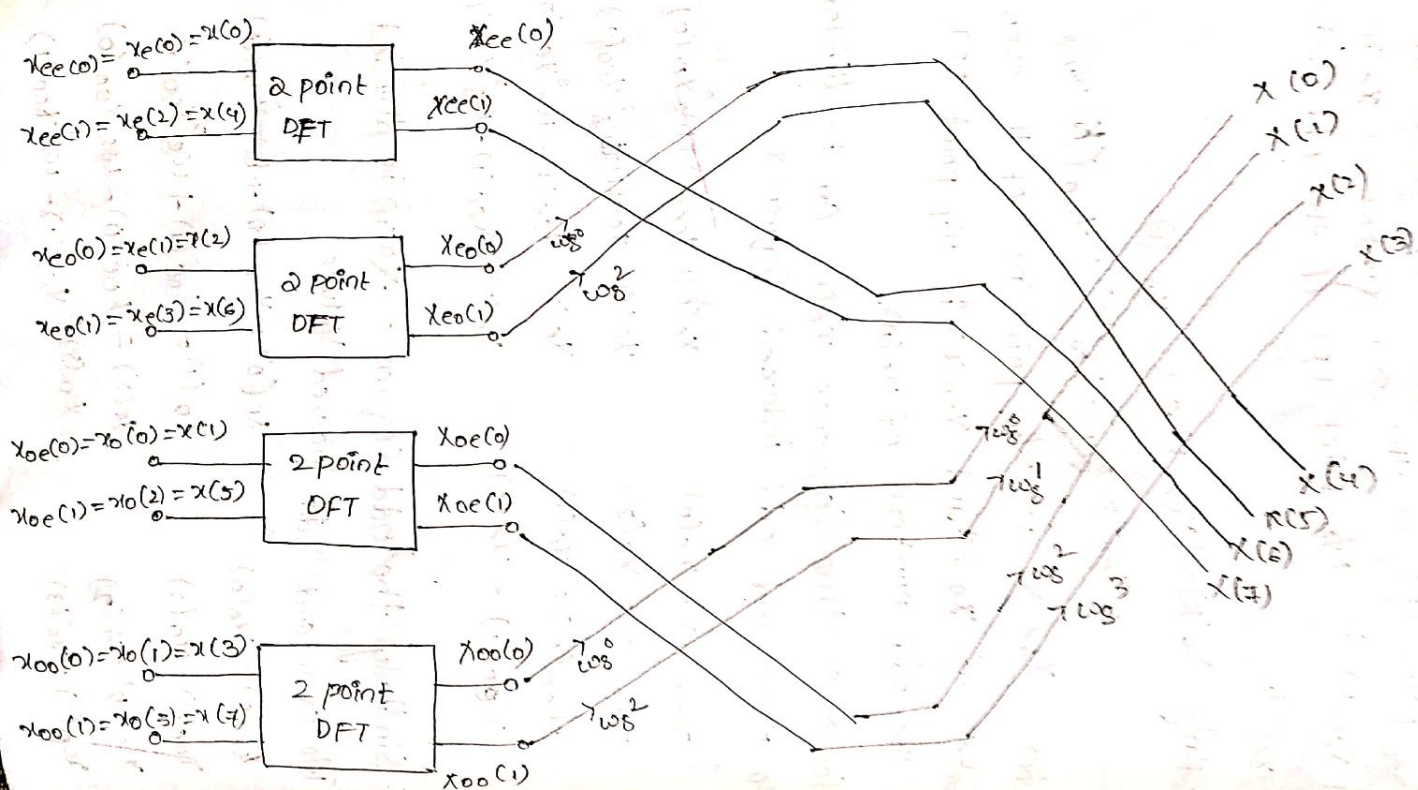
$$x_{oo}(1) = x_o(3)$$

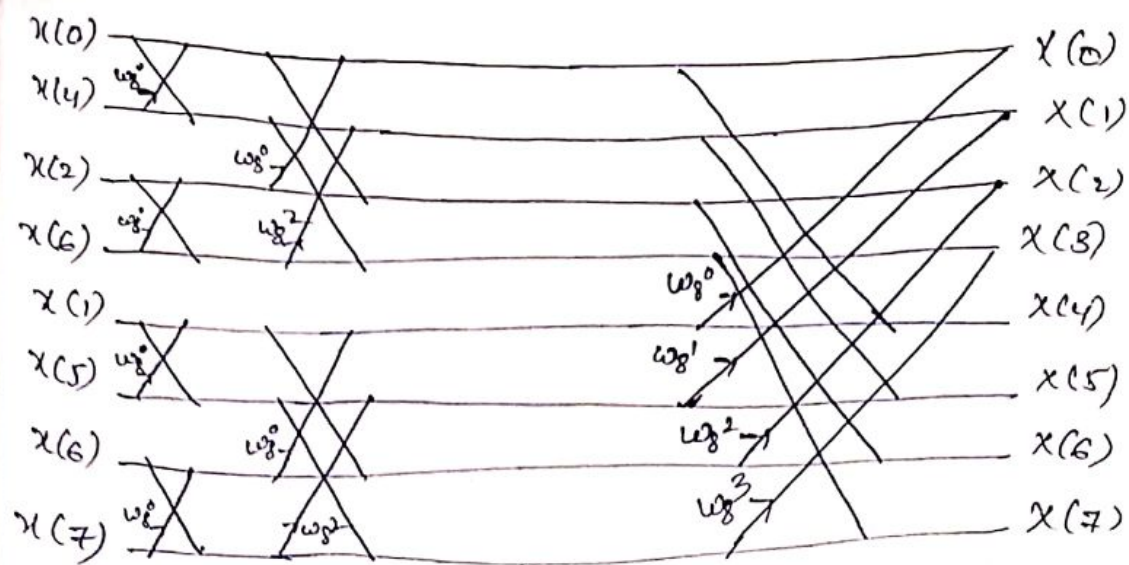
$$X_o(0) = X_{oe}(0) + W_8^0 X_{oo}(0)$$

$$X_o(1) = X_{oe}(1) + W_8^2 X_{oo}(1)$$

$$X_o(2) = X_{oe}(0) - W_8^0 X_{oo}(0)$$

$$X_o(3) = X_{oe}(1) - W_8^2 X_{oo}(1)$$





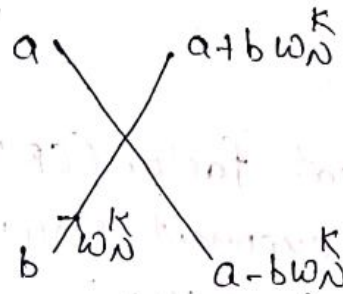
i/p Sample	Binary rep of i/p Sample	Bit reversal Binary	Bit Reversal sample Index
0	0 0 0	0 0 0	0
1	0 0 1	1 0 0	4
2	0 1 0	0 1 0	2
3	0 1 1	1 1 0	6
4	1 0 0	0 0 1	5
5	1 0 1	1 0 1	3
6	1 1 0	0 1 1	7
7	1 1 1	1 1 1	1

In DFT Algorithm we can find that the output Sequences to be in natural order that $X(k)$ where $k=0$ to $m-1$. the input sequence has to be stored in a bit reversal order for an 8-point DFT Algorithm the input sequence is in order $x(0), x(4), x(2), x(6), x(1), x(5), x(3), x(7)$.

for $n=8$ the bit Reversal process is shown in the table.

Basic operation:-

The Basic operation of DIT Algorithm is represented by using butterfly in which two inputs are combined to give two outputs at the input a and b are the given values and the output are shown in the figure.



Steps of Radix-2 Decimation in time FFT:-

1. The no. of i/p samples $N = 2^M$, where 'M' is an integer.
2. The i/p sequence is shuffled through bit reversal.
3. The no. of stages in the flow graph is given by $M = \log_2 N$.
4. Each stage consists of $N/2$ butterflies.
5. i/p or outputs for each butterfly are separated by 2^{m-1} , where 'm' represents the stage index. That is for first stage $m=1$, & for second stage $m=2$, and so on.
6. The no. of complex multiplications is given by $\frac{N}{2} \log_2 N$.
7. The no. of complex additions is given by $N \log_2 N$.

8. The twiddle factor exponents are the function of stage 'm' index and is given by

$$K = \frac{Nt}{2^m}, \quad t = 0, 1, \dots, 2^{m-1}$$

9. The no of sets of butterflies in each stage given by 2^{M-m}

10. The exponent Repeat factor (ERF) which is the no of times the exponent sequence associated with 'm' is repeated by 2^{M-m}

1) Find DFT of Sequence $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$ using decimation in time algorithm.

Sol $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$

$$N = 8$$

$$M = \log_2 N = \log_2 2^3 = 3$$

$$\omega_8^0 = (e^{-j2\pi/8})^0 = 1$$

$$\omega_8^1 = (e^{-j2\pi/8})^1$$

$$= e^{-j\pi/4} = \cos \pi/4 - j \sin \pi/4 = 1/\sqrt{2} - j 1/\sqrt{2}$$

$$= 0.707 - 0.707j$$

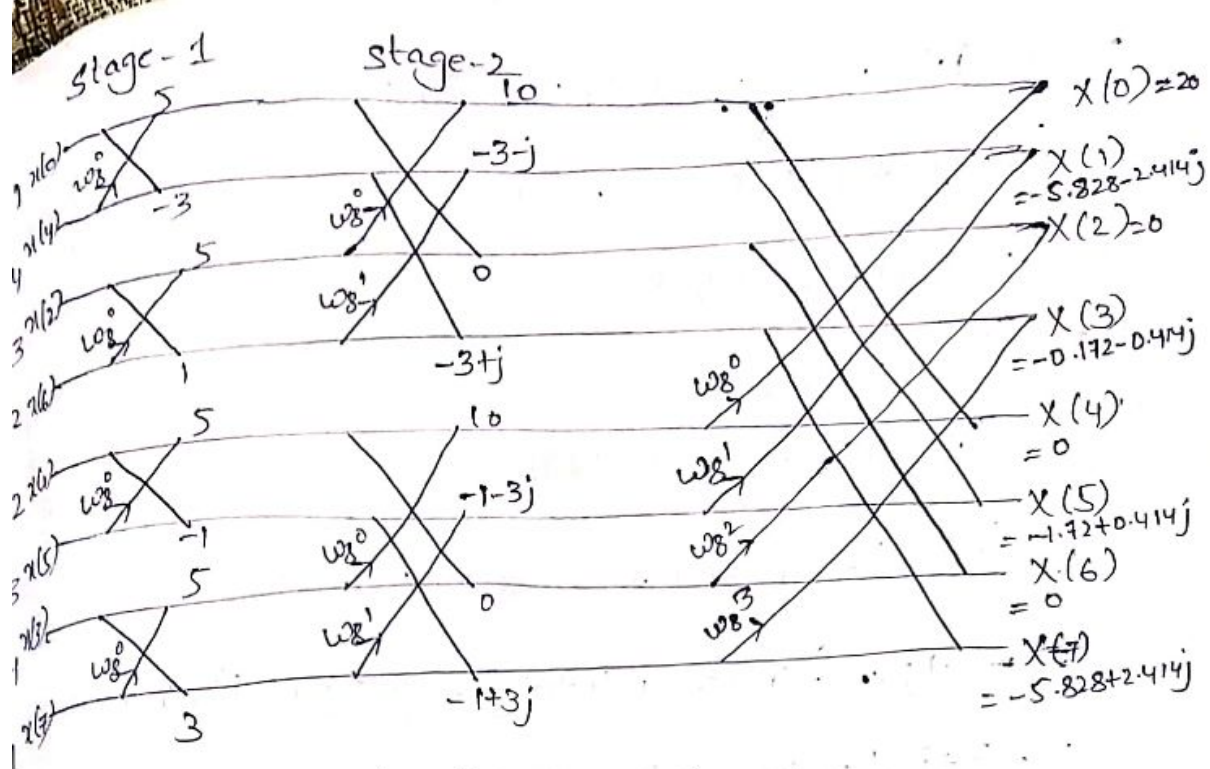
$$\omega_8^2 = (e^{-j2\pi/8})^2$$

$$= e^{-j2\pi/8 \cdot 2} = e^{-j\pi/2} = \cos \pi/2 - j \sin \pi/2 = 0 - j = -j$$

$$\omega_8^3 = (e^{-j2\pi/8})^3$$

$$= e^{-j2\pi \cdot 3/8} = e^{-j3\pi/4}$$

$$= \cos 3\pi/4 - j \sin 3\pi/4 = -0.707 - 0.707j$$



$$X(1) = (-3-j) + (0.707 - 0.707j)(-1-3j)$$

$$= -3-j - 0.707 + j0.707 - 3 \times j0.707 + 3j^2 \times 0.707$$

$$= -3-j - 0.707 + j0.707 - 2.12j - 2.12$$

$$= -5.83 - j(-1 + 0.707 - 2.12)$$

$$= -5.83 - 2.41j$$

$$X(5) = (-3-j) - (0.707 - j0.707)(-1+3j)$$

$$= -3-j - (-0.707 + j0.707 - 3 \times 0.707j + 3j^2 \times 0.707)$$

$$= -3-j + 0.707 - j0.707 + 3j \times 0.707 + 3 \times 0.707$$

$$= -3-j + 0.707 - 0.707j + 2.12j + 2.12$$

$$= -0.17 - j(1 + 0.707 - 2.12)$$

$$= -0.17 + 0.41j$$

$$X(2) = 0 + w_8^2(x(0)) = 0$$

$$X(6) = 0 - w_8^2(x(0)) = 0$$

$$X(3) = (-3+j) + w_8^3(-1+3j)$$

$$= (-3+j) + (-0.707 + j0.707)(-1+3j)$$

$$= -3 + j + 0.707 - 0.707j - 2.12j - 2.12$$

$$= -4.41 + j(1 - 0.707 - 2.12)$$

$$= -4.41 + j(-1.83)$$

$$= -4.41 - 1.83j$$

$$X(7) = (-3 + j) - \omega_8^3(-1 + 3j)$$

$$= (-3 + j) - (-0.707 + j0.707)(-1 + 3j)$$

$$= -3 + j + (-0.707 - 0.707j - 2.12j + 3j^2 \times 0.707)$$

$$= -3 + j + 0.707 + 0.707j + 2.12j + 2.12$$

$$= -5.828 + j2.414$$

$$X(k) = \{ 20, -5.828 - 2.414j, 0, -0.172 - j0.414, 0, -1.72 + j0.414, 0, -5.828 + j2.414 \}$$

2) Compute the 8-point DFT of the sequence

$$x(n) = \begin{cases} 1, & 0 \leq n \leq 7 \\ 0, & \text{otherwise} \end{cases} \text{ using decimation in time.}$$

Sol $\omega_8^0 = (e^{-j2\pi/8})^0 = 1$

$$\omega_8^1 = (e^{-j2\pi/8})^1 = \cos \pi/4 - j \sin \pi/4 = 0.707 - j0.707$$

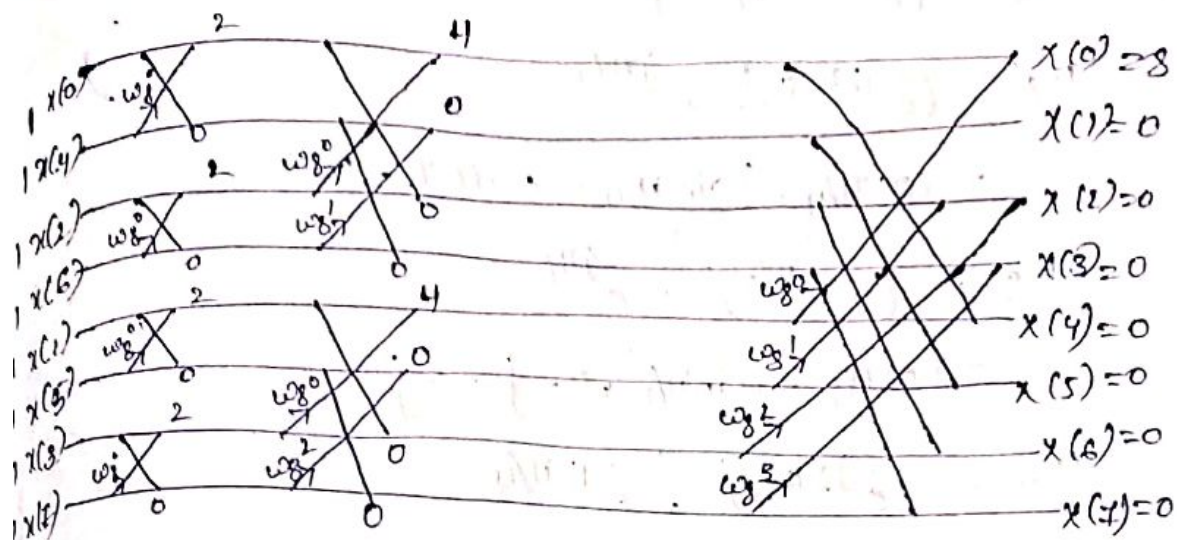
$$\omega_8^2 = (e^{-j2\pi/8})^2 = \cos \pi/2 - j \sin \pi/2 = 0 - j = -j$$

$$\omega_8^3 = (e^{-j2\pi/8})^3 = \cos 3\pi/4 - j \sin 3\pi/4 = -0.707 - j0.707$$

$$N = 8$$

$$M = \log_2 N = \log_2 8 = 3$$

$$L = \log_2 2^3 = 3$$

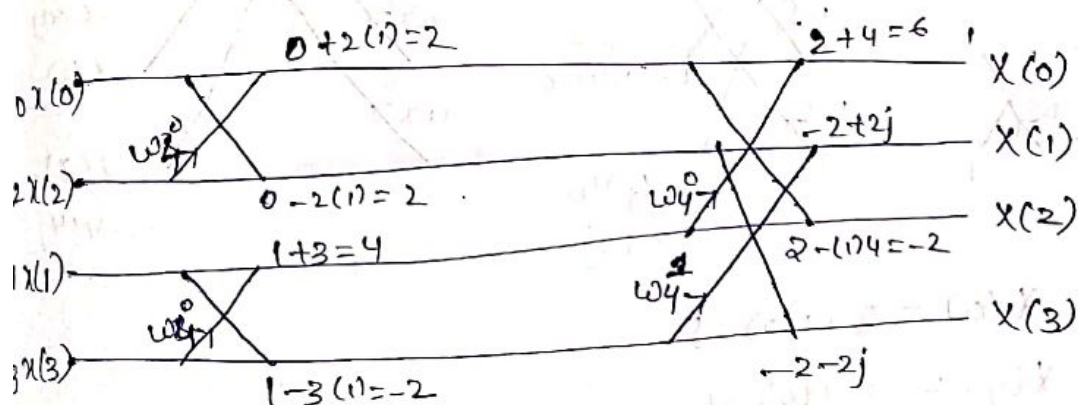


$$X(k) = \{8, 0, 0, 0, 0, 0, 0, 0\}$$

3) Compute 4-point DFT of the sequence $x(n) = \{0, 1, 2, 3\}$ using decimation in time Algorithm.

$$w_4^0 = (e^{-j2\pi/4})^0 = 1$$

$$w_4^1 = (e^{-j2\pi/4})^1 = e^{-j\pi/2} = \cos\pi/2 - j\sin\pi/2 = 0 - j = -j$$



$$X(k) = \{6, -2+2j, -2, -2-2j\}$$

4) Find 8-point DFT of Sequence $x(n) = \{1, 1, 1, 1, 1, 0, 0, 0\}$ using Decimation in time algorithm.

$$\omega_8^0 = (e^{-j2\pi/8})^0 = 1$$

$$\omega_8^1 = (e^{-j2\pi/8})^1 = e^{-j\pi/4}$$

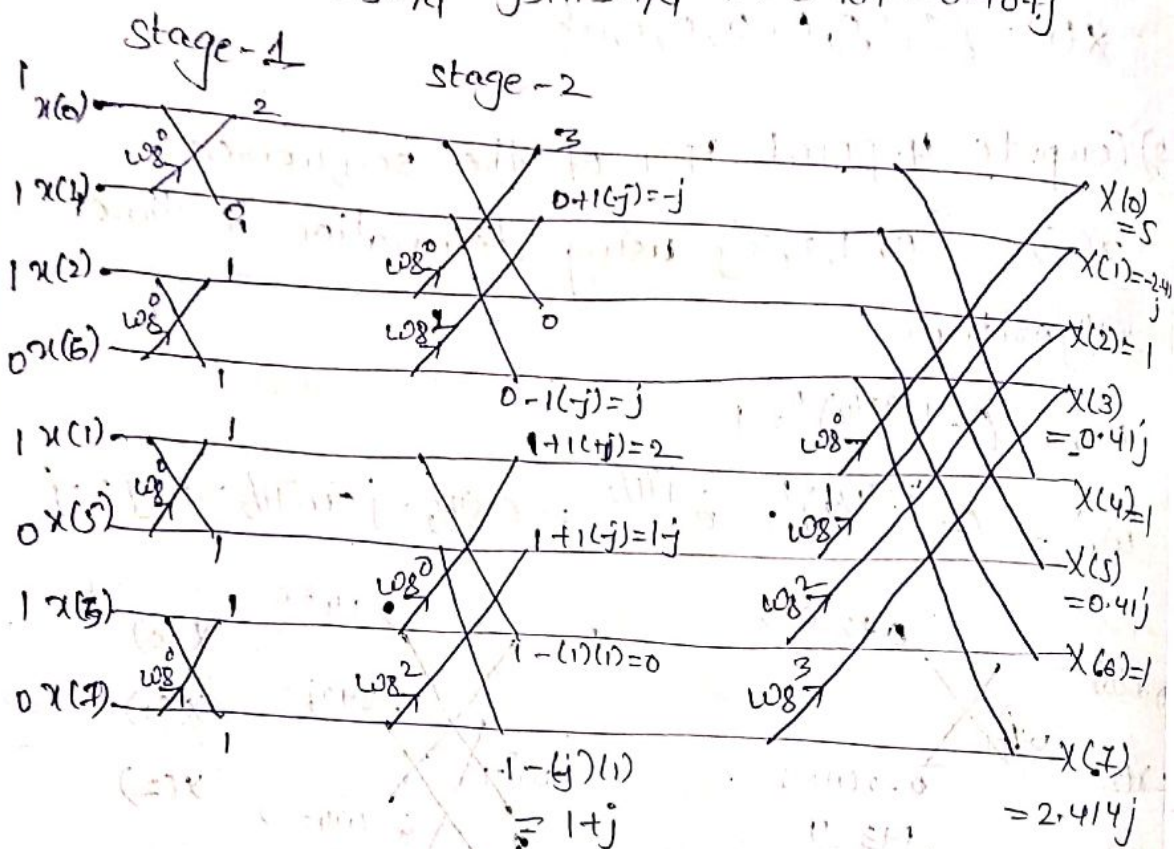
$$= \cos \pi/4 - j \sin \pi/4 = 0.707 - 0.707j$$

$$\omega_8^2 = (e^{-j2\pi/8})^2 = e^{-j\pi/2}$$

$$= \cos \pi/2 - j \sin \pi/2 = 0 - j = -j$$

$$\omega_8^3 = (e^{-j2\pi/8})^3 = e^{-j3\pi/4}$$

$$= \cos 3\pi/4 - j \sin 3\pi/4 = -0.707 - 0.707j$$



$$X(0) = 3 + 1(2) = 5$$

$$X(4) = 3 - 1(2) = 1$$

$$X(1) = -j + (0.707 - j0.707)(1-j)$$

$$= -j + (0.707 - j0.707 - 0.707j + j^2 0.707)$$

$$= -j + 0.707 - 1.41j - 0.707$$

$$= -2.41j$$

$$X(5) = -j - (0.707 - 0.707j)(1+j)$$

$$= -j - (0.707 - 0.707j - 0.707j + 0.707j^2)$$

$$= -j - (0.707 - 1.41j - 0.707)$$

$$= -j + 1.41j = 0.41j$$

$$X(2) = 1 + (j)(0) = 1$$

$$X(6) = 1 - (j)(0) = 1$$

$$X(3) = j + (-0.707 - 0.707j)(1+j)$$

$$= j + (-0.707 - 0.707j - 0.707j - j^2 0.707)$$

$$= j + (-0.707 - 1.41j + 0.707)$$

$$= j - 1.41j = -0.41j$$

$$X(7) = j - (-0.707 - j 0.707)(1+j)$$

$$= j - (-0.707 - 0.707j - 0.707j - j^2 0.707)$$

$$= j - (-0.707 - 1.41j + 0.707)$$

$$= j + 1.41j$$

$$= 2.41j$$

$$X(k) = \{5, -2.41j, 1, -0.41j, 1, 0.41j, 1, 2.41j\}$$

⑤ Draw the flow graph & the DIT and FFT

No. of Input Samples $N = 16$

$$N = 16 = 2^4$$

→ The DIT sequence bit Reversal

$$N = 16 = 2^4$$

0 - 0000 → 0000 - 0	9 - 1001 → 1001 - 9
1 - 0001 → 1000 - 8	10 - 1010 → 0101 - 5
2 - 0010 → 0100 - 4	11 - 1011 → 1101 - 3
3 - 0011 → 1100 - 12	12 - 1100 → 0011 - 3
4 - 0100 → 0010 - 2	13 - 1101 → 1011 - 11
5 - 0101 → 1010 - 10	14 - 1110 → 0111 - 7
6 - 0110 → 0110 - 6	15 - 1111 → 1111 - 15
7 - 0111 → 1110 - 14	
8 - 1000 → 0001 - 1	

Steps

No. of stages $M = \log_2 N$

$$= \log_2 16 = 4$$

Each stage have $\frac{N}{2}$ butterflies $\Rightarrow \frac{16}{2} = 8$

spacing b/w butterfly $\times 2^{n-1} = 2^0 = 1$

→ No. of complex multiplication $S = \frac{N}{2} \log_2 N$

$$= \frac{16}{2} \log_2 16 = 8 \log_2 16$$

$$= 8 \log_2 2^4$$

$$= 8 \times 4 \log_2 2 = 32$$

→ Twiddle factor

$$W_N^k = \frac{N!}{2^m}$$

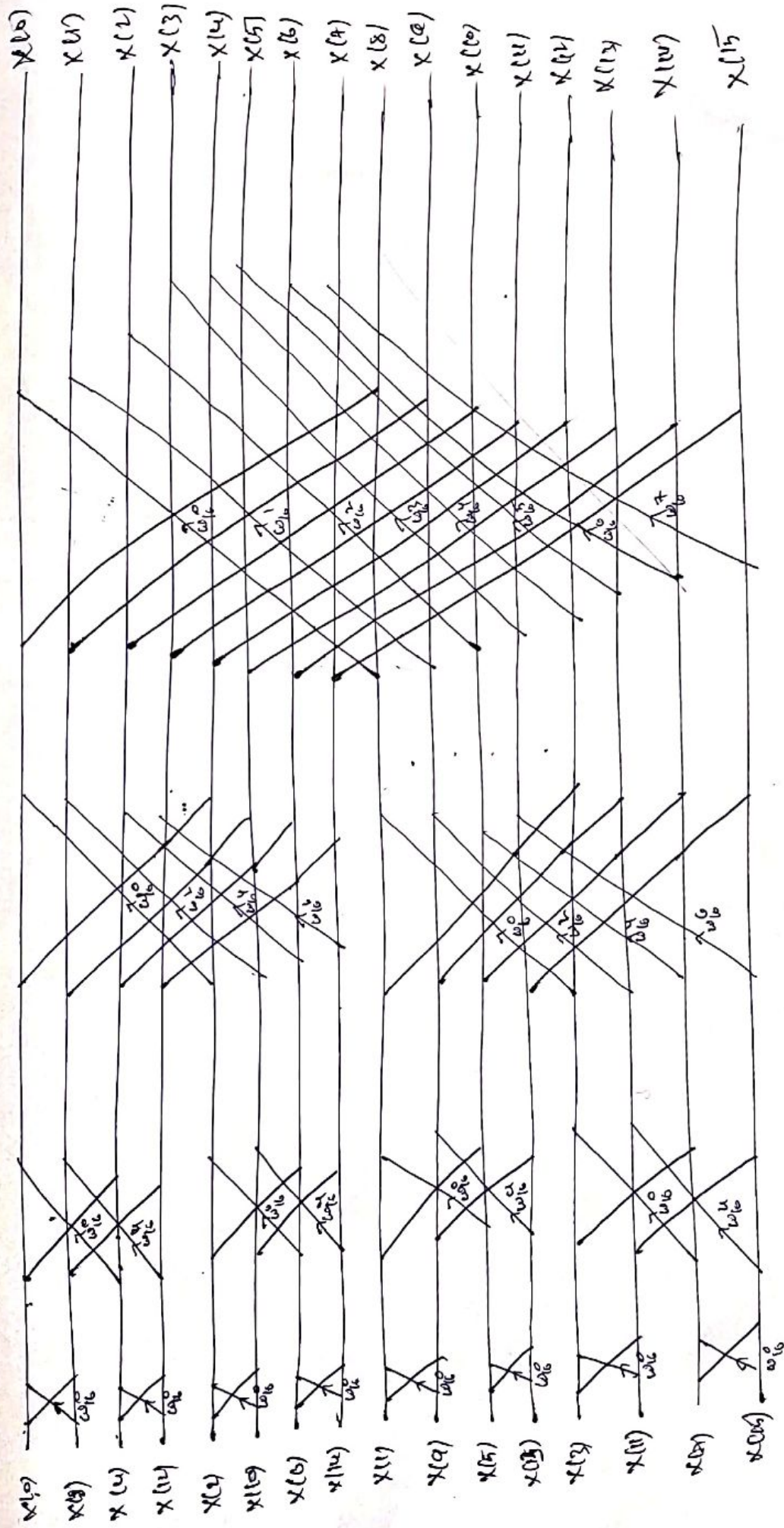
$$= \frac{16(0)}{2}$$

$$k=0$$

$$t \rightarrow 0 \text{ to } 2^{m-1} - 1 \quad m=1$$

$$0 \text{ to } 2^0 - 1$$

$$= 0 \text{ to } 0$$



Transform Algorithm:-

Decimation in Frequency Fast Fourier Transform

In Decimation in frequency Algorithm the output Sequence $x(k)$ is divided into 2^{th} smaller and smaller Sub Sequence.

In this Algorithm the input sequence $x(n)$ is partitioned into two Sequence each of length $N/2$ samples. the first Sequence $x_1(n)$ consists of first $N/2$ samples of $x(n)$. and Second Sequence $x_2(n)$ consists of the last $N/2$ samples of $x(n)$.

i.e., $x_1(n) = x(n)$ for $n = 0, 1, \dots, N/2 - 1$

$x_2(n) = x(n + N/2)$ for $n = 0, 1, \dots, N/2 - 1$

The N -point DFT of $x(n)$ can be written as

$$X(k) = \sum_{n=0}^{N-1} x(n) \omega_N^{kn}$$

$$= \sum_{n=0}^{N/2-1} x(n) \omega_N^{kn} + \sum_{n=N/2}^{N-1} x(n) \omega_N^{kn}$$

$$= \sum_{n=0}^{N/2-1} x_1(n) \omega_N^{kn} + \sum_{n=0}^{N/2-1} x_2(n) \omega_N^{k(n+N/2)}$$

$$= \sum_{n=0}^{N/2-1} x_1(n) \omega_N^{kn} + \omega_N^{kN/2} \sum_{n=0}^{N/2-1} x_2(n) \omega_N^{kn}$$

$$\begin{aligned} \omega_N^{kN/2} &= e^{-j \frac{2\pi}{N} \frac{kN}{2}} \\ &= e^{-j\pi k} \end{aligned}$$

$$X(k) = \sum_{n=0}^{N/2-1} x_1(n) \omega_N^{kn} + e^{-j\pi k} \sum_{n=0}^{N/2-1} x_2(n) \omega_N^{kn}$$

for Even Values of k $e^{-j\pi k} = 1$

$$\therefore X(2k) = \sum_{n=0}^{N/2-1} x_1(n) \omega_N^{kn} + \sum_{n=0}^{N/2-1} x_2(n) \omega_N^{kn}$$

$$= \sum_{n=0}^{N/2-1} [x_1(n) + x_2(n)] \omega_N^{2kn}$$

$$= \sum_{n=0}^{N/2-1} f(n) \omega_N^{2kn}$$

where $f(n) = x_1(n) + x_2(n)$

* $f(n)$ is obtained by adding the first half and last half of the input sequence $x(n)$.

When k is odd $e^{-j\pi k} = -1$

$$\therefore X(2k+1) = \sum_{n=0}^{N/2-1} x_1(n) \omega_N^{kn} - \sum_{n=0}^{N/2-1} x_2(n) \omega_N^{kn}$$

$$= \sum_{n=0}^{N/2-1} [x_1(n) - x_2(n)] \omega_N^{(2k+1)n}$$

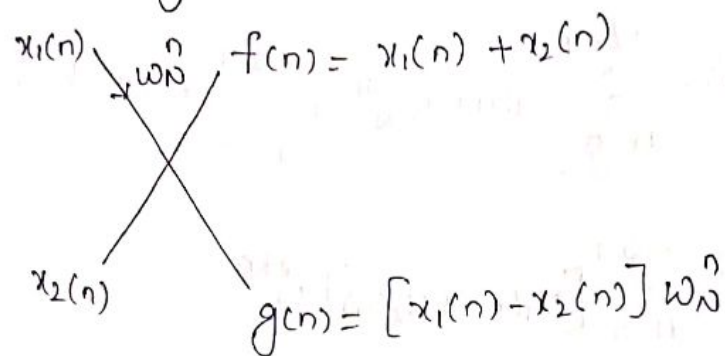
$$= \sum_{n=0}^{N/2-1} [x_1(n) - x_2(n)] \omega_N^n \omega_N^{2kn}$$

$$= \sum_{n=0}^{N/2-1} g(n) \omega_N^{2kn}$$

where $g(n) = [x_1(n) - x_2(n)] \omega_N^n$

* $g(n)$ is obtained by subtracting the second half of the sequence from the first half of the input sequence and then multiplying the results with the relevant sequence ω_N^n .

from the above two sequences of $f(n)$ and $g(n)$, the basic operation can be represented by butterfly as shown in figure.



For $n=8$ we have $X(0) = \sum_{n=0}^{N/2-1} f(n) \omega_N^{2kn}$

$$X(0) = \sum_{n=0}^3 f(n) \omega_8^{2(0)n}$$

$$= \sum_{n=0}^3 f(n) \cdot 1$$

$$= f(0) + f(1) + f(2) + f(3)$$

$$X(2) = \sum_{n=0}^3 f(n) \omega_8^{2(1)n}$$

$$= \sum_{n=0}^3 f(n) \omega_8^{2n}$$

$$= f(0) \omega_8^{2(0)} + f(1) \omega_8^{2(1)} + f(2) \omega_8^{2(2)} + f(3) \omega_8^{2(3)}$$

$$= f(0) + f(1) \omega_8^2 + f(2) \omega_8^4 + f(3) \omega_8^6$$

$$\omega_8^4 = -1$$

$$\omega_8^8 = 1$$

$$\omega_8^6 = -\omega_8^2$$

$$X(2) = f(0) + f(1) \omega_8^2 - f(2) - f(3) \omega_8^2$$

$$X(4) = \sum_{n=0}^3 f(n) \omega_8^{2(2)n}$$

$$X(4) = \sum_{n=0}^3 f(n) \omega_8^{4n}$$

$$= \sum_{n=0}^3 f(n) (-1)^n$$

$$= f(0) (-1)^0 + f(1) (-1)^1 + f(2) (-1)^2 + f(3) (-1)^3$$

$$X(4) = f(0) - f(1) + f(2) - f(3)$$

$$X(6) = \sum_{n=0}^3 f(n) \omega_8^{2(3)n}$$

$$= \sum_{n=0}^3 f(n) \omega_8^{6n}$$

$$= \sum_{n=0}^3 f(n) (-\omega_8^2)^n$$

$$= f(0) (-\omega_8^2)^0 + f(1) (-\omega_8^2)^1 + f(2) (-\omega_8^2)^2 +$$

$$f(3) (-\omega_8^2)^3$$

$$= f(0) - f(1) \omega_8^2 - f(2) + f(3) \omega_8^2$$

$$X(1) = \sum_{n=0}^{N/2-1} g(n) \omega_N^{2kn}$$

$$= \sum_{n=0}^3 g(n) \omega_8^{2(0)n}$$

$$= \sum_{n=0}^3 g(n) \cdot 1$$

$$= g(0) + g(1) + g(2) + g(3)$$

$$X(3) = \sum_{n=0}^3 g(n) \omega_8^{2(1)n}$$

$$= \sum_{n=0}^3 g(n) \omega_8^{2n}$$

$$= g(0) \omega_8^{2(0)} + g(1) \omega_8^{2(1)} + g(2) \omega_8^{2(2)} + g(3) \omega_8^{2(3)}$$

$$X(3) = g(0) + g(1) \omega_8^2 + g(2) \omega_8^4 + g(3) \omega_8^6$$

$$X(3) = g(0) + g(1) \omega_8^2 - g(2) - g(3) \omega_8^2$$

$$X(5) = \sum_{n=0}^3 g(n) \omega_8^{2(3)n}$$

$$= \sum_{n=0}^3 g(n) \omega_8^{6n}$$

$$= \sum_{n=0}^3 g(n) (-1)^n$$

$$= g(0) (-1)^0 + g(1) (-1)^1 + g(2) (-1)^2 + g(3) (-1)^3$$

$$= g(0) - g(1) + g(2) - g(3)$$

$$X(7) = \sum_{n=0}^3 g(n) \omega_8^{2(3)n}$$

$$= \sum_{n=0}^3 g(n) \omega_8^{6n}$$

$$= \sum_{n=0}^3 g(n) (\omega_8^2)^n$$

$$= g(0) (-\omega_8^2)^0 + g(1) (-\omega_8^2)^1 + g(2) (-\omega_8^2)^2 +$$

$$g(3) (-\omega_8^2)^3$$

$$= g(0) - g(1) \omega_8^2 + g(2) (-1) - g(3) (\omega_8^6)$$

$$= g(0) - g(1) \omega_8^2 - g(2) + g(3) \omega_8^2$$

We have seen that the even indexed samples of $X(k)$ can be obtained from the four point DFT of the sequences $f(n)$. where $f(n) = x_1(n) + x_2(n)$.

$$f(0) = x_1(0) + x_2(0)$$

$$f(1) = x_1(1) + x_2(1)$$

$$f(2) = x_1(2) + x_2(2)$$

$$f(3) = x_1(3) + x_2(3)$$

The odd indexed values of $X(k)$ can be obtained from the 4-point DFT of Sequence $g(n)$.

$$g(n) = [x_1(n) - x_2(n)] \omega_N^n$$

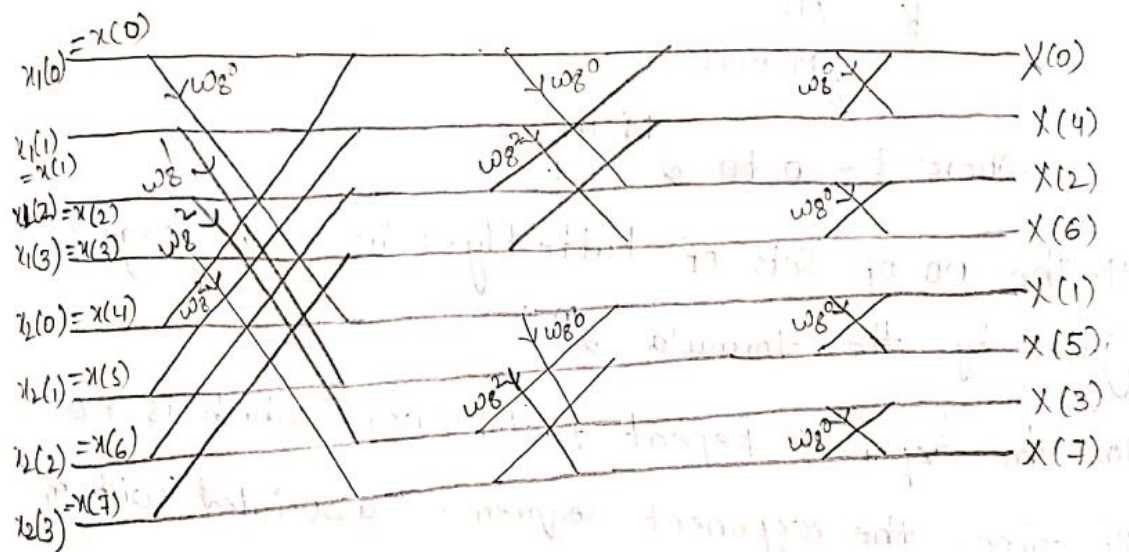
$$g(n) = [x_1(n) - x_2(n)] \omega_8^n$$

$$g(0) = [x_1(0) - x_2(0)] \omega_8^0$$

$$g(1) = [x_1(1) - x_2(1)] \omega_8^1$$

$$g(2) = [x_1(2) - x_2(2)] \omega_8^2$$

$$g(3) = [x_1(3) - x_2(3)] \omega_8^3$$



Steps for Radix-2 Decimation in Frequency FFT

Algorithm:-

1. The no of input Samples $N = 2^M$, where M is the no of stages.

2. The input Sequence is in natural order.

3. The no of stages in the flow graph is given by

$$M = \log_2 N$$

4. Each stage consists of $N/2$ butterflies.
5. inputs (or) output for each butterfly are separated by 2^{M-m} samples.

where 'm' represents the stage index that is for first stage $M=1$ and second

6. The no of complex multiplications is given by $N/2 \log_2 N$.

7. The no of complex additions is given by $N \log_2 N$

8. The twiddle factor components (or) a function of stage index and is given by

$$k = \frac{Nt}{2^{M-m+1}}$$

Where $t = 0$ to $2^{M-m} - 1$

9. The no of sets of butterflies in each stage is given by the formula 2^{m-1}

10. The exponent Repeat factor (ERF) which is no of times the exponent Sequence associated with 'm' repeated is given by $2^m - 1$

Difference And Similarities Between DIT and DIF

Difference

1. For Decimation in time the input is bit reversal while the output is natural order. Where as for Decimation in frequency the i/p is in natural order while the output is bit Reversal order.
2. The DIF butterfly is ^{Slightly} different from DIT where in DIF the complex multiplication takes place after the add Subtract operation.

Similarities:-

1. Both Algorithms require $N \log_2 N$ operations to calculate the DFT.
2. Both Algorithms can be done in place and both need to perform bit Reversal at some place during the calculation.

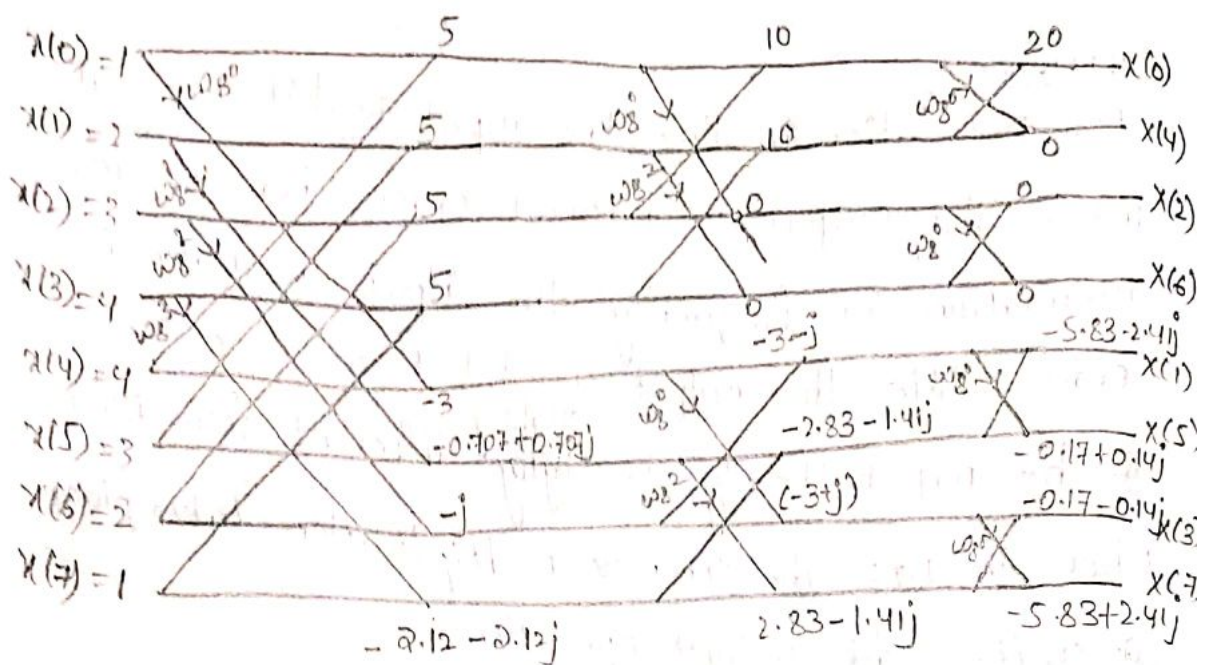
1) Find the DFT of the Sequence $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$ using Decimation in frequency Algorithm.

Sol: given $\omega_8^0 = (e^{-j2\pi/8})^0 = 1$

$$\omega_8^1 = 0.707 - j0.707$$

$$\omega_8^2 = -j$$

$$\omega_8^3 = -0.707 - j0.707$$



$$(-3+j)(1)$$

$$-0.707 + 0.707j - 2.12 - 2.12j$$

$$-3-j + 2.83 - 1.41j$$

$$-5.83 - 2.41j$$

$$-0.707 + 0.707j + 2.12 + 2.12j$$

$$-3-j + 2.83 + 1.41j$$

$$(1.41 + 2.83j)(-j)$$

$$(0.17 + 0.41j)$$

$$-1.41j + 2.83$$

$$-3+j + 2.83 - 1.41j$$

$$-0.17 - 0.41j$$

$$(-3+j - 2.83 + 1.41j)$$

$$X(k) = \{ 20, -5.83 - 2.41j, 0, -0.17 - 0.14j, 0, -0.17 + 0.14j, 0, -5.83 + 2.41j \}$$

2) Find the Calculate the 8-point DFT of the Sequence $x(n) = \begin{cases} 1, & 0 \leq n \leq 7 \\ 0, & \text{otherwise} \end{cases}$ by using Decimation in time & Decimation in frequency Algorithms.

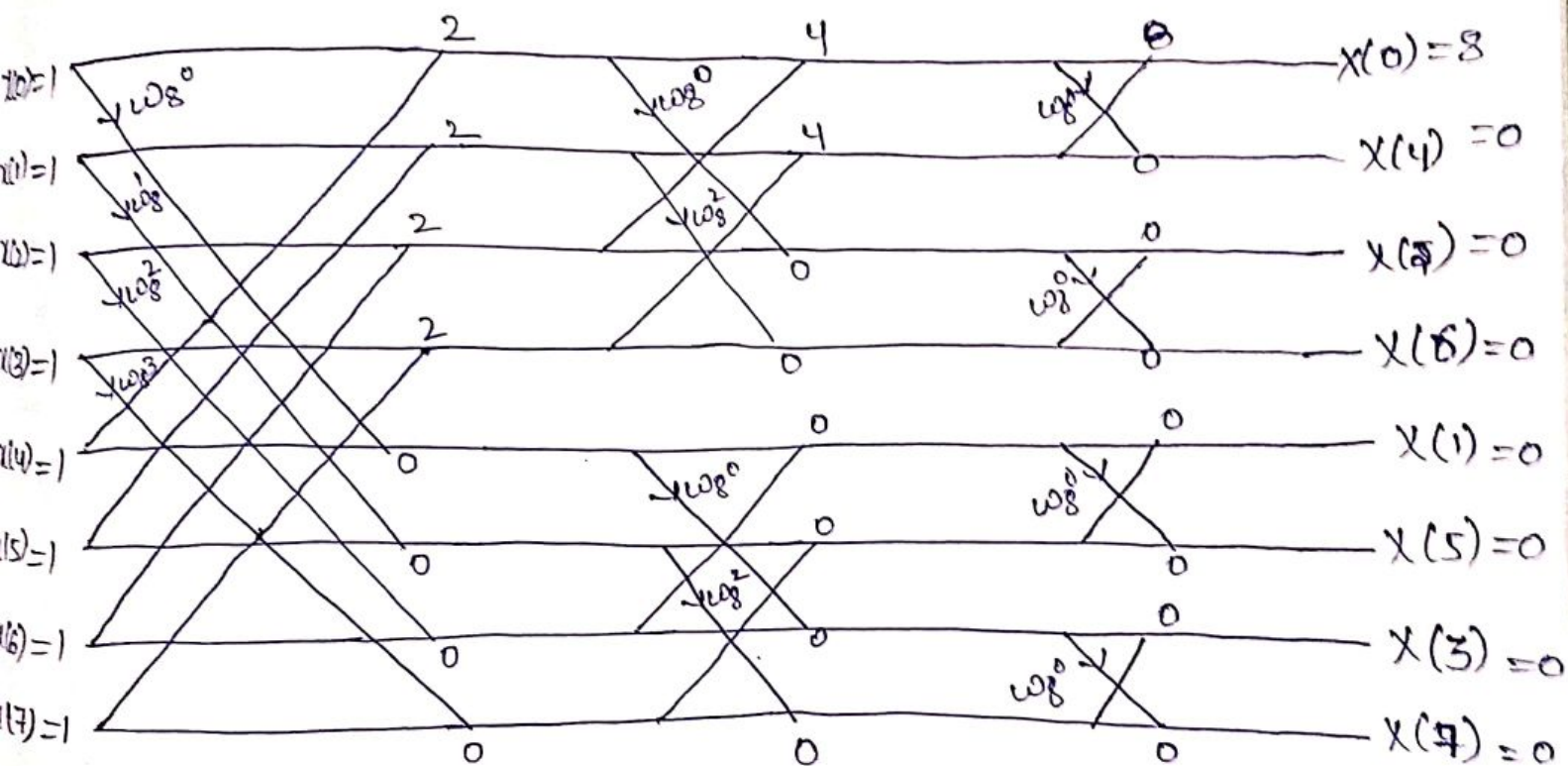
sol) Decimation in Frequency

$$\omega_8^0 = 1$$

$$\omega_8^1 = 0.707 - j0.707$$

$$\omega_8^2 = -j$$

$$\omega_8^3 = -0.707 - j0.707$$



$$X(k) = \{8, 0, 0, 0, 0, 0, 0, 0\}$$

i) Review of Z-Transforms2. Realization of Digital filter structures:

Digital filters can be realized by the following two realization techniques:

1. Recursive Realization. (or) IIR Digital filter structures.
2. Non-Recursive Realization (or) FIR Digital filter structures.

1. Recursive Realization:

for recursive realization the current o/p $y(n)$ is a function of present i/p, past i/p and past o/p samples.

$$y(n) = \{ x(n), x(n-1), x(n-2), \dots \text{ \& } y(n-1), y(n-2), \dots \}$$

This realization corresponds to IIR Digital filter structures.

2. Non Recursive Realization:

for Non Recursive realization, the current o/p $y(n)$ is a function of present & past i/p samples only. This realization corresponds to FIR Digital filter structures.

$$y(n) = \{ x(n), x(n-1), x(n-2), x(n-3), \dots \}$$

IR Digital filter structures (or) Recursive Realization techniques:

There are seven well known to realize IR Digital filter.

- i) Direct form - I
- ii) Direct form - II (canonic) min. no. of memory locations required realization technique.
- iii) Transposed Realization.
- iv) parallel cascade Realization.
- v) parallel Realization.
- vi) Lattice Realization
- vii) Ladder Realization.

i) Direct form - I Realization technique:

Let us consider a system (IR Digital filter) described by a N^{th} order LCCDE is given by
(linear constant coefficient difference eqn)

$$y(n) = -\sum_{k=1}^N a_k y(n-k) + \sum_{k=0}^M b_k x(n-k) \quad \text{--- (1)}$$

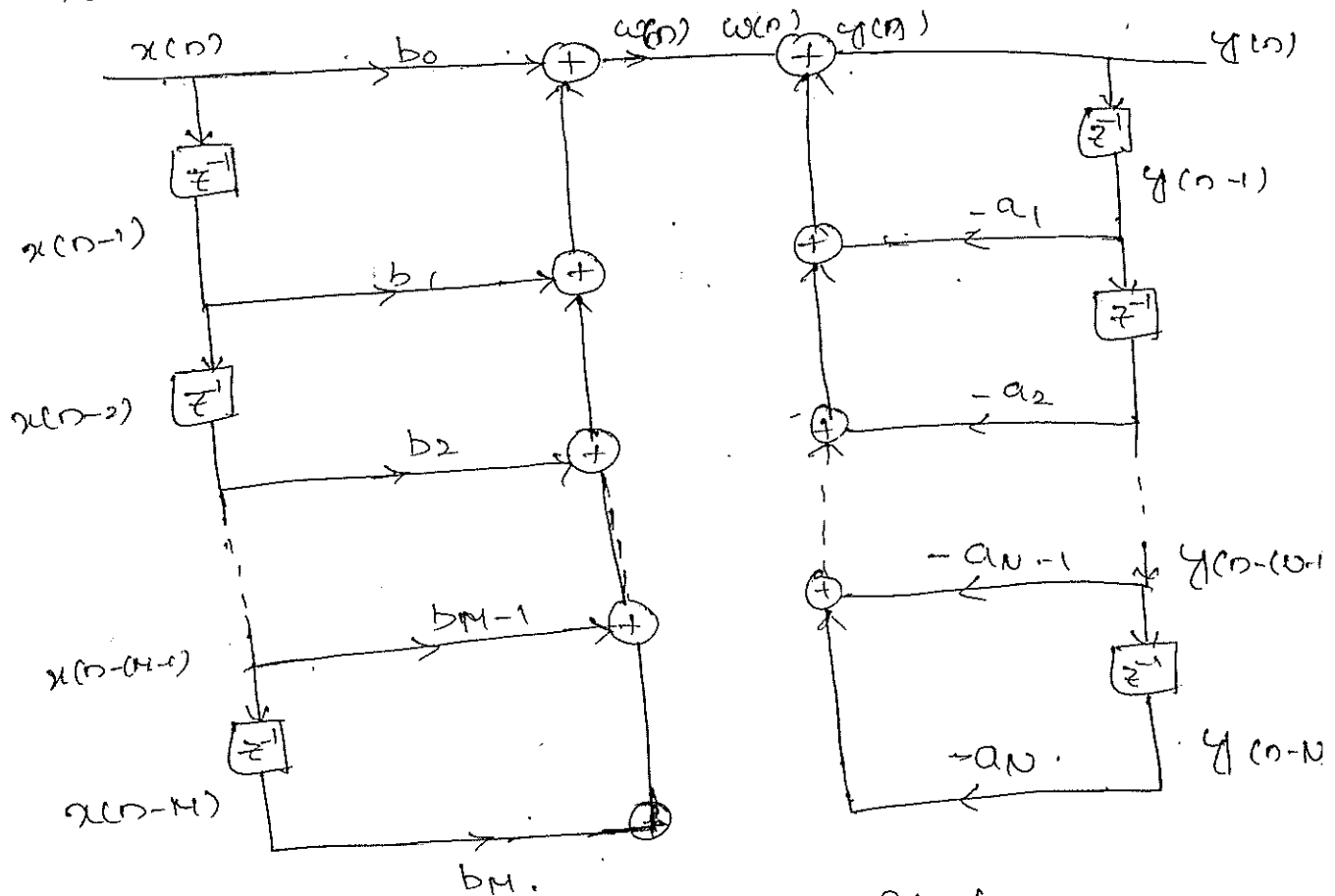
$$\begin{aligned} y(n) = & -a_1 y(n-1) - a_2 y(n-2) - \dots - a_{N-1} y(n-(N-1)) - a_N y(n-N) + b_0 x(n) + \\ & b_1 x(n-1) + b_2 x(n-2) + \dots + b_{M-1} x(n-(M-1)) \\ & + b_M x(n-M) \quad \text{--- (2)} \end{aligned}$$

consider $w(n) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \dots + b_{M-1} x(n-(M-1)) + b_M x(n-M)$

--- (3)

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) - \dots - a_{N-1} y(n-N+1) - a_N y(n-N) + w(n) \quad \text{--- (4)}$$

Eqn (3) can be realized by using direct form 2 realization as shown in fig(a).



fig(a).

fig(b).

Eqn (4) can be realized as shown in fig(c).
 fig(c): Direct Form-II structure

The above structure is called Direct form - I NR Digital filter structure, which used separate delays for both input and output as shown in above fig.

This realization requires, total no. of multiplications = $M+N+1$.

The total no. of additions required = $M+N$.

The total no. of memory ^{locations} required = $M+N$.

Direct Form II (or) canonic (or) Minimum no. of memory locations realization:

Let us consider a system (i.e. digital filter) described by N^{th} order difference equation:

$$y(n) = - \sum_{k=1}^N a_k y(n-k) + \sum_{k=0}^M b_k x(n-k) \quad \text{--- (1)}$$

Apply z -Transform.

$$Y(z) = - \sum_{k=1}^N a_k Y(z) z^{-k} + \sum_{k=0}^M b_k X(z) z^{-k}$$

$$Y(z) \cdot \left[1 + \sum_{k=1}^N a_k z^{-k} \right] = \sum_{k=0}^M b_k X(z) z^{-k}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{1 + \sum_{k=1}^N a_k z^{-k}}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}$$

$$\frac{Y(z)}{W(z)} = \sum_{k=0}^M b_k z^{-k}$$

$$Y(z) = W(z) [b_0 z^0 + b_1 z^{-1} + b_2 z^{-2} + \dots]$$

$$y(n) = w(n)b_0 + w(n-1)b_1 + \dots \quad \text{--- (2)}$$

$$\frac{W(z)}{X(z)} = \frac{1}{1 + \sum_{k=1}^N a_k z^{-k}}$$

if N
+ N.

$$W(z) \left[1 + \sum_{k=1}^N a_k z^{-k} \right] = X(z).$$

$$W(z) \neq X(z)$$

$$W(z) + \sum_{k=1}^N a_k z^{-k} W(z) = X(z)$$

Div.

Apply $z \cdot T$, we get

$$z \cdot T \left[W(z) + \sum_{k=1}^N a_k z^{-k} W(z) \right] = z \cdot T [X(z)]$$

By linearity property of $z \cdot T$, we get

$$z \cdot T [W(z)] + \sum_{k=1}^N a_k z \cdot T [z^{-k} W(z)] = z \cdot T [X(z)]$$

we know:

$$x(n) \xleftrightarrow{z \cdot T} X(z)$$

$$w(n) \longleftrightarrow W(z)$$

$$w(n-k) \longleftrightarrow z^{-k} W(z)$$

$$y(n) \longleftrightarrow Y(z)$$

$$W(z) + \sum_{k=1}^N a_k z^{-k} W(z) = X(z)$$

$$W(z) = X(z) - \sum_{k=1}^N a_k z^{-k} W(z)$$

$$w(n) = x(n) - a_1 w(n-1) - a_2 w(n-2) - \dots - a_{N-1} w(n-(N-1)) - a_N w(n-N) \quad \text{--- (1)}$$

$$w(n-(N-1)) - a_N w(n-N) \quad \text{--- (1)}$$

$$\frac{Y(z)}{W(z)} = \sum_{k=0}^M b_k z^{-k}$$

$$Y(z) = \sum_{k=0}^M b_k z^{-k} W(z)$$

Apply IFT & linearity property

$$\mathcal{F}^{-1} \left[Y(z) \right] = \sum_{k=0}^M b_k \mathcal{F}^{-1} \left[z^{-k} \omega(z) \right]$$

$$y(n) = \sum_{k=0}^M b_k \omega(n-k)$$

$$y(n) = b_0 \omega(n) + b_1 \omega(n-1) + b_2 \omega(n-2) + \dots + b_{m-1} \omega(n-(m-1)) + b_m \omega(n-m) \quad \text{--- (1)}$$

eqn (1) can be realized using direct form-I structure as shown in below figs

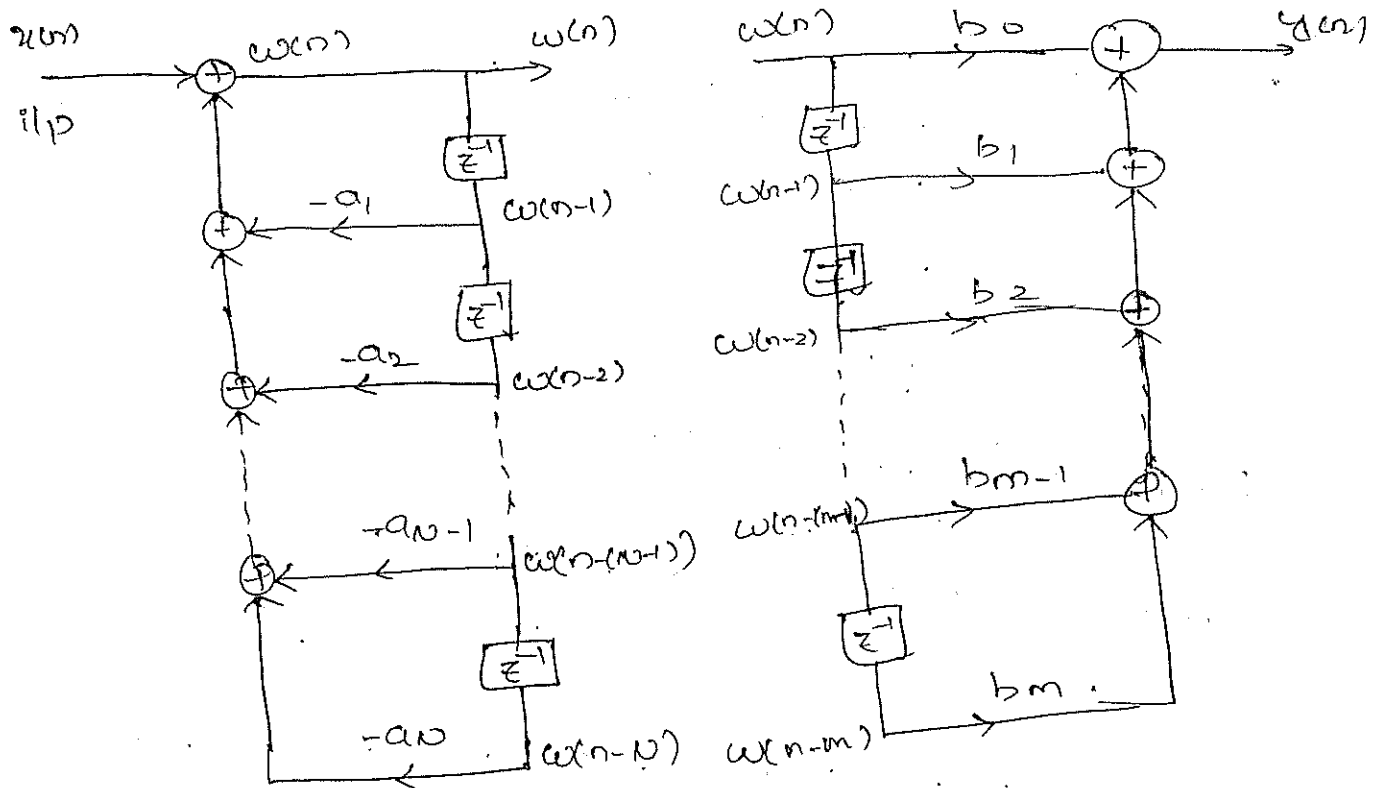
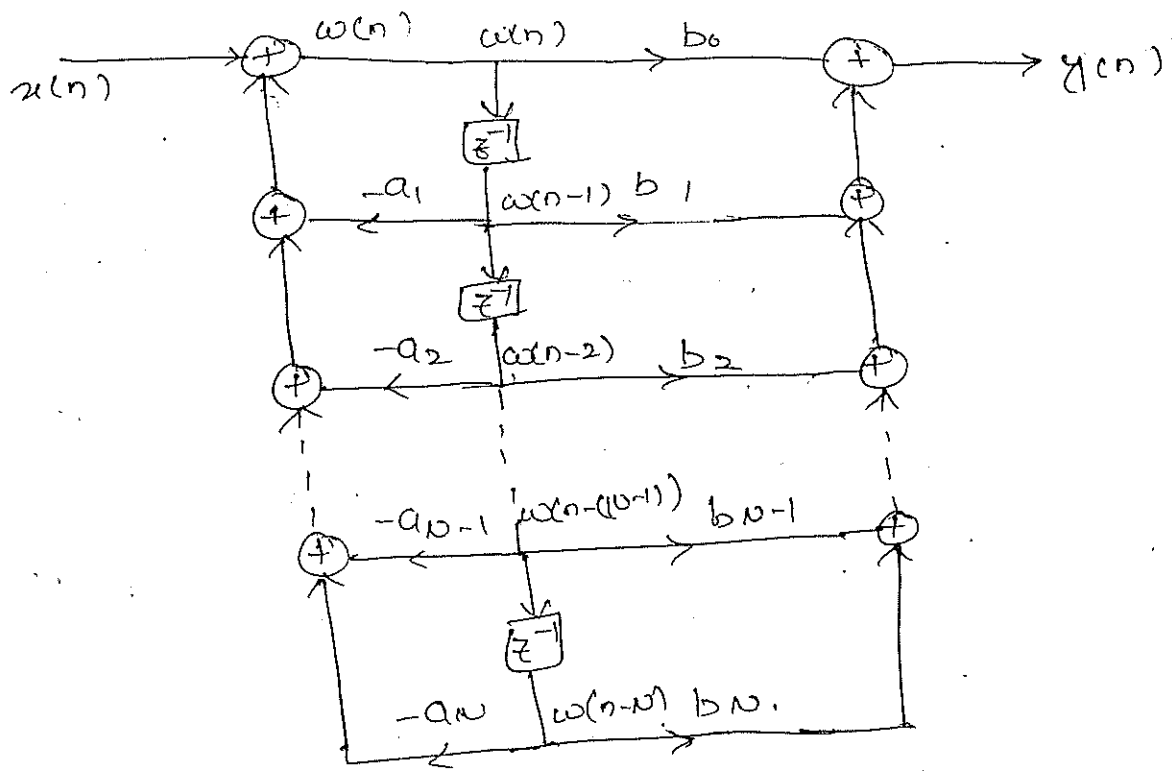


Fig. a

Eqn (2) can be realized as show in fig. b.

for $N=m$



N^{th} Order LCCDE

fig (a): Direct form-II structure for N^{th} Order LCCDE

The above structure is called Direct Form-II structure, which used common delay factors both i/p & o/p side. By using this realization, we can reduce half of delay terms (or) memory locations. Compare with direct form-I structure. Therefore, this realization is also known as Canonical (or) min. no. of memory locations required realization. For this realization, the total no. of multipliers required $= M+N$.

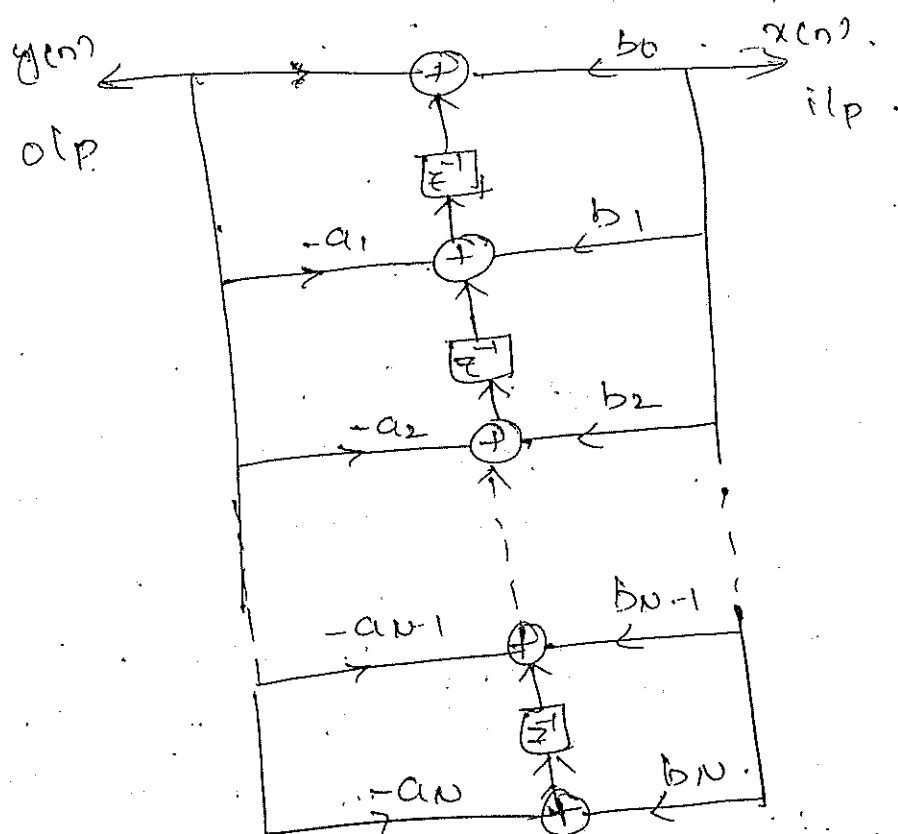
total no. of adders $= M+N$.

total no. of memory locations $= \max(M, N)$.

(ii) Transposed structure:

Procedure steps:

1. from the given difference eqn, we have to implement direct form-II structure.
2. Interchange ilps and o/p
3. Reverse the direction of flow of signals in the direct form II structure.
4. Interchange nodes and summing points



iv) Cascade Realization Technique:

Cascade Realization is implemented by attaching all the factors of system function $H(z)$ in series that is cascade. Let us consider the Nth order system is described by system function

$$H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{1 + \sum_{k=1}^N a_k z^{-k}}$$

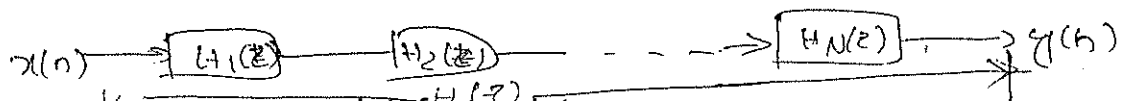
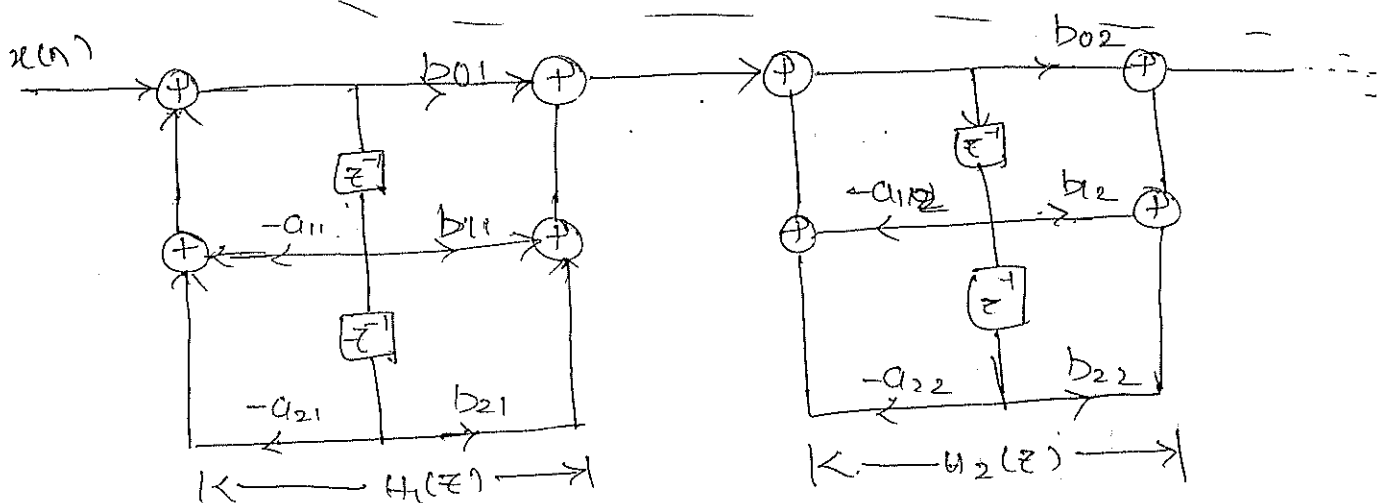
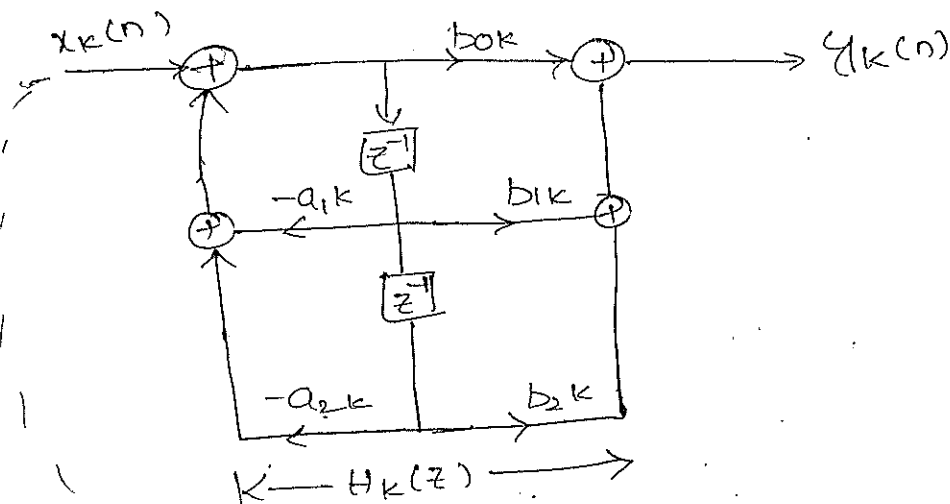
Let the system function $H(z)$ is represented in series of subsystems, each subsystem's highest degree is '2'.

i.e., $H(z) = \prod_{k=1}^N H_k(z) = H_1(z) \cdot H_2(z) \cdots H_N(z)$

Here, N - Integer part of $\left(\frac{N+1}{2}\right)$

π - product of subsystem of $H(z)$

$$H_k(z) = \frac{b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2}}{1 + a_{1k} z^{-1} + a_{2k} z^{-2}}$$



v) parallel form realization technique:

parallel form realization is obtained by making use of partial fraction expansion of system function $H(z)$.

let us consider N^{th} order system which is described by $H(z)$.

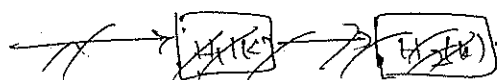
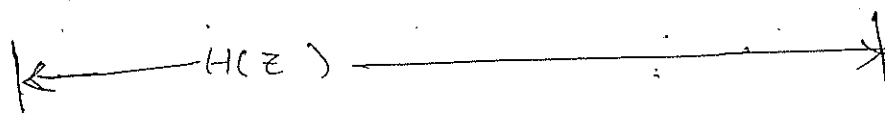
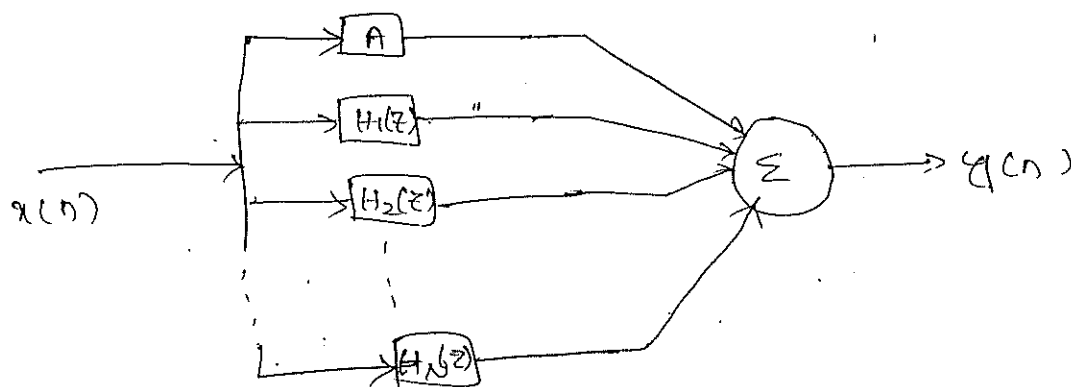
i.e.,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^m b_k z^{-k}}{1 + \sum_{k=1}^N a_k z^{-k}} = A + \sum_{k=1}^N \frac{A_k}{1 - a_k z^{-1}}$$

$$= A + \sum_{k=1}^N H_k(z)$$

$$H(z) = A + \sum_{k=1}^N H_k(z) = A + H_1(z) + H_2(z) + \dots + H_N(z)$$

Here, $A = \frac{b_m}{a_N}$

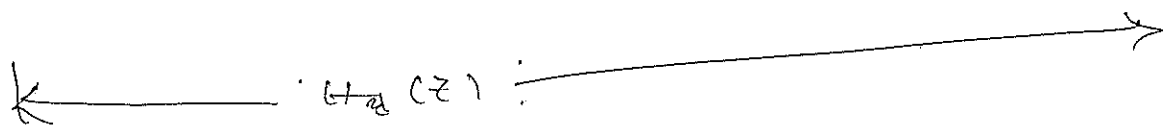
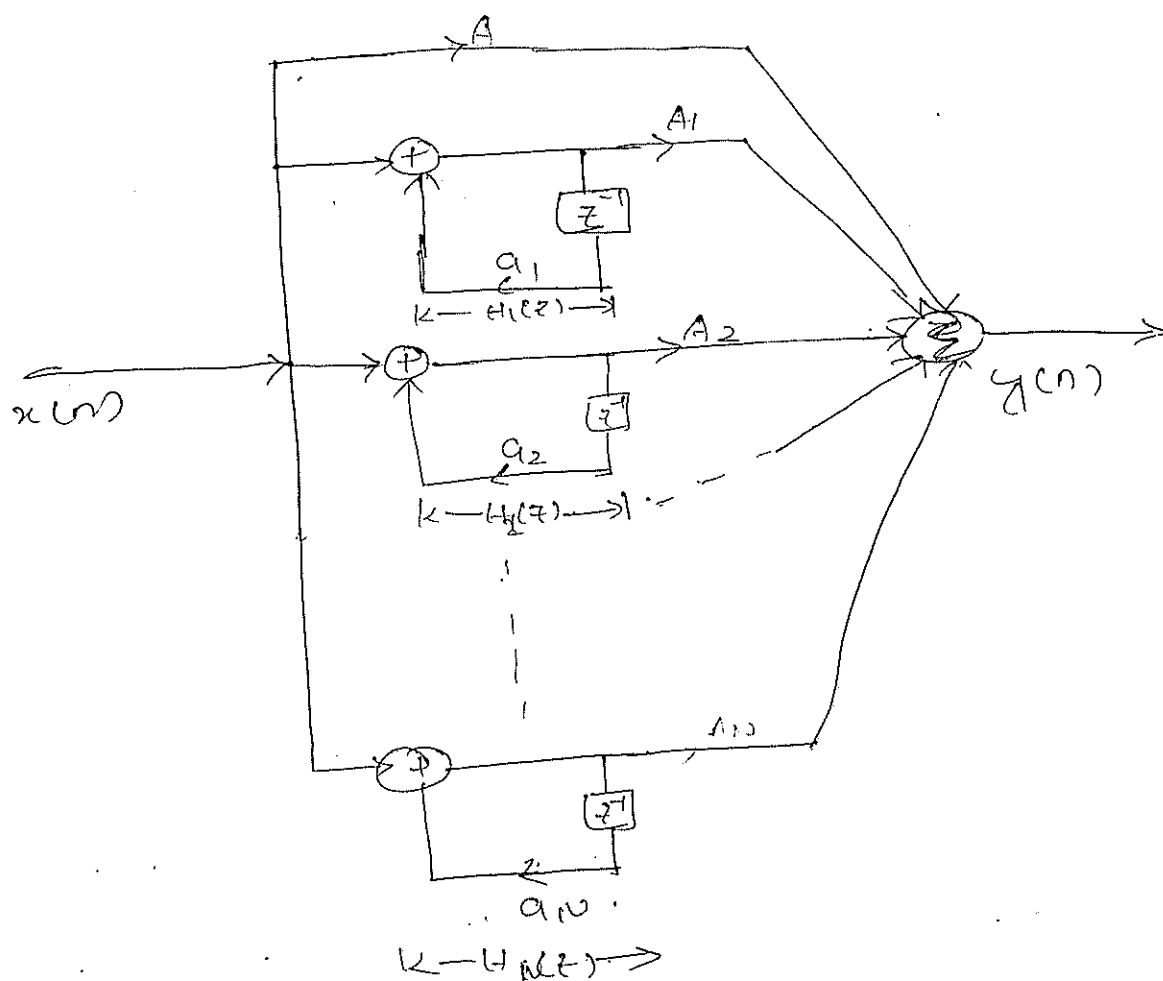
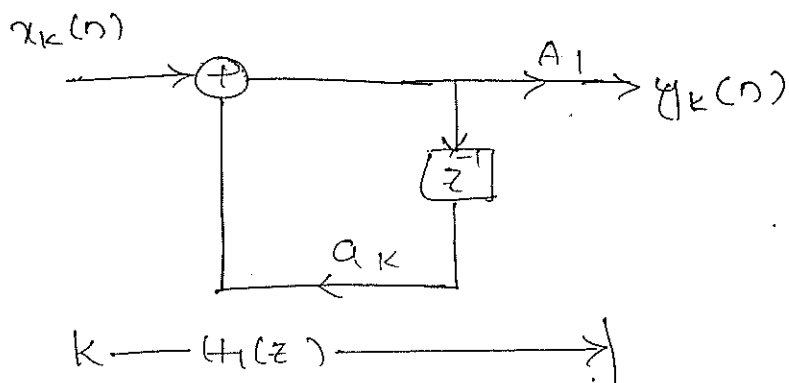


ham-
action

3m

$u(z)$

$$H_k(z) = \frac{A_k}{1 - a_k z^{-1}}$$



Problems:

1. Realize the following systems.

$$i. \quad y(n) = -\frac{3}{8}y(n-1) + \frac{3}{32}y(n-2) + \frac{1}{64}y(n-3) + x(n) + 3x(n-1) + 2x(n-2) \quad \text{by using}$$

i) Direct form-I, Direct form-II & transposed structure.

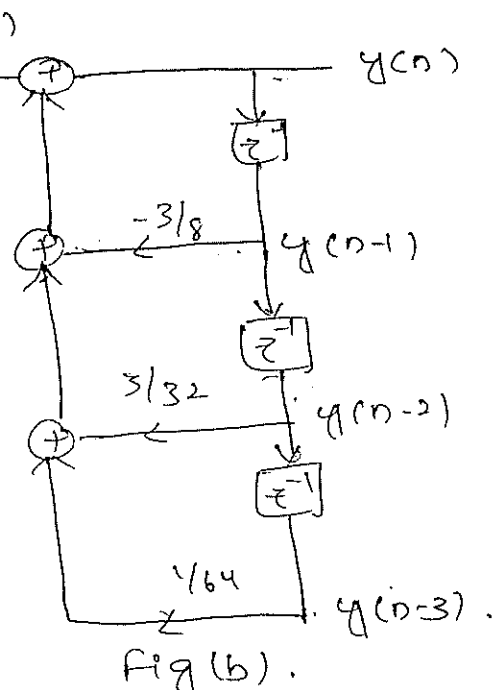
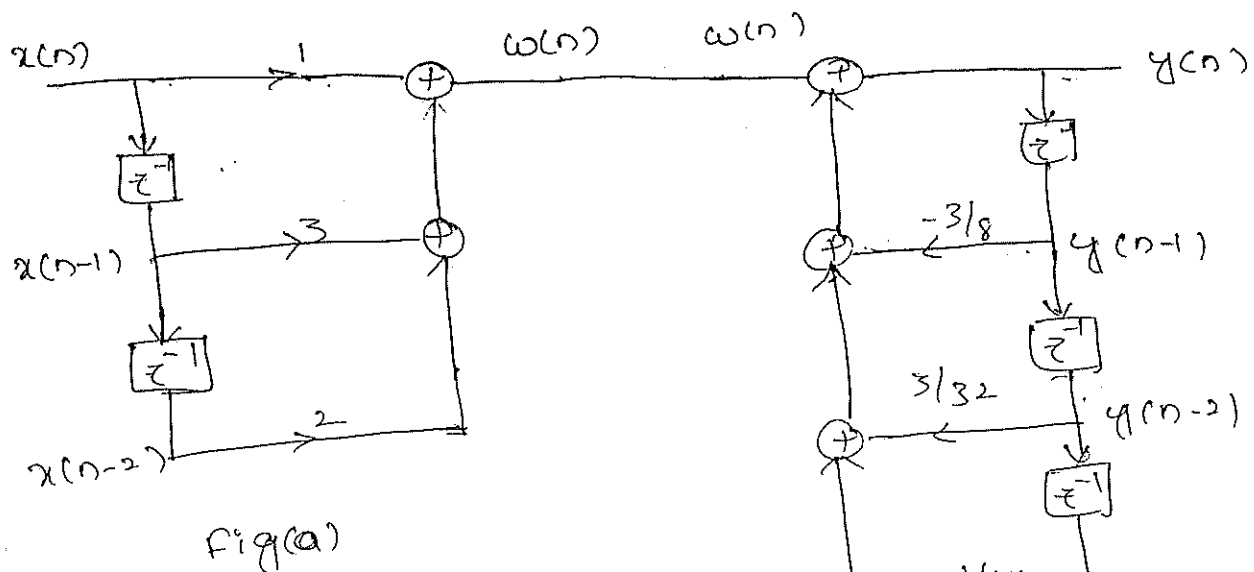
i) Direct form I

$$y(n) = -\frac{3}{8}y(n-1) + \frac{3}{32}y(n-2) + \frac{1}{64}y(n-3) + x(n) + 3x(n-1) + 2x(n-2)$$

Let $w(n) = x(n) + 3x(n-1) + 2x(n-2)$, then

$$y(n) = -\frac{3}{8}y(n-1) + \frac{3}{32}y(n-2) + \frac{1}{64}y(n-3) + w(n) \quad \text{--- (1)}$$

Eqn can be realized as shown in fig(a).
Eqn (2) as shown in fig(b)



Fig(c)

This realization requires

$$\begin{aligned}\text{Total no. of multipliers} &= M+N+1 \\ &= 2+3+1 \\ &= 6\end{aligned}$$

$$\begin{aligned}\text{Summers} &= M+N \\ &= 2+3 = 5\end{aligned}$$

$$\begin{aligned}\text{Memory location} &= M+N \\ &= 2+3 \\ &= 5\end{aligned}$$

Direct form II :-

$$\begin{aligned}y(n) &= -\frac{3}{8}y(n-1) + \frac{3}{32}y(n-2) + \frac{1}{64}y(n-3) + x(n) \\ &\quad + 3x(n-1) + 2x(n-2)\end{aligned}$$

Applying z.T on both sides, we get

$$\begin{aligned}Y(z) &= -\frac{3}{8}z^{-1}Y(z) + \frac{3}{32}z^{-2}Y(z) + \frac{1}{64}z^{-3}Y(z) + X(z) \\ &\quad + 3z^{-1}X(z) + 2z^{-2}X(z)\end{aligned}$$

$$Y(z) \left[1 + \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3} \right] = X(z) \left[1 + 3z^{-1} + 2z^{-2} \right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 3z^{-1} + 2z^{-2}}{1 + \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3}}$$

$$= \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}$$

$$\frac{W(z)}{X(z)} = \frac{1}{1 + \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3}}$$

$$\omega(z) \left[1 + \frac{3}{8} z^{-1} - \frac{3}{32} z^{-2} - \frac{1}{64} z^{-3} \right] = x(z)$$

$$\omega(z) = x(z) - \frac{3}{8} z^{-1} \omega(z) + \frac{3}{32} z^{-2} \omega(z) - \frac{1}{64} z^{-3} \omega(z)$$

Applying Inverse z -Transform.

$$\omega(n) = x(n) - \frac{3}{8} \omega(n-1) + \frac{3}{32} \omega(n-2) - \frac{1}{64} \omega(n-3)$$

— ①

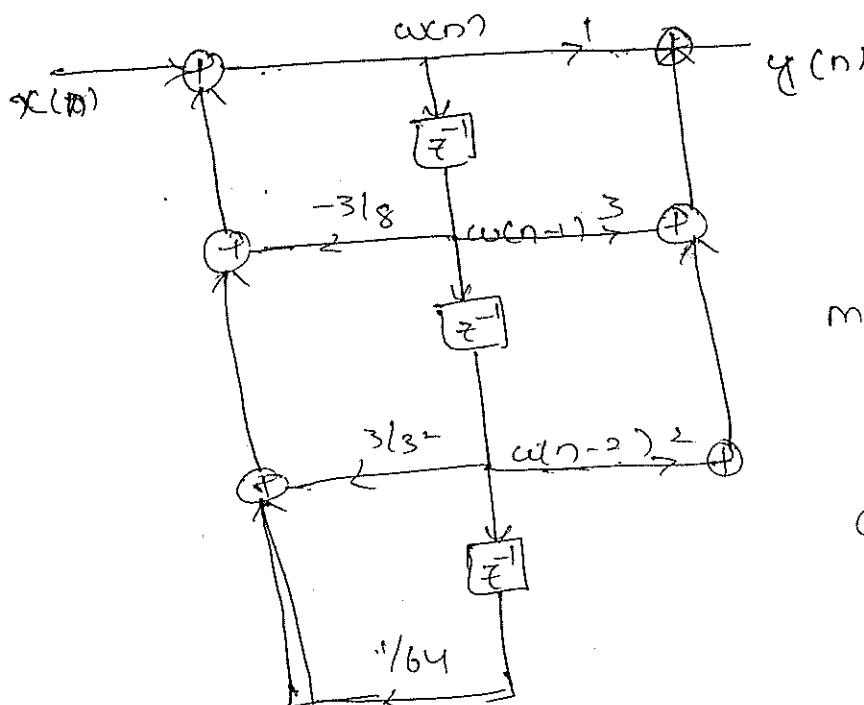
$$\frac{y(z)}{\omega(z)} = 1 + \frac{3}{8} z^{-1} - \frac{3}{32} z^{-2} + \frac{1}{64} z^{-3}$$

$$y(z) = \left[1 + \frac{3}{8} z^{-1} - \frac{3}{32} z^{-2} + \frac{1}{64} z^{-3} \right] \omega(z)$$

$$= \omega(z) + \frac{3}{8} z^{-1} \omega(z) - \frac{3}{32} z^{-2} \omega(z) + \frac{1}{64} z^{-3} \omega(z)$$

Apply IZT

$$y(n) = \omega(n) + \frac{3}{8} \omega(n-1) - \frac{3}{32} \omega(n-2) + \frac{1}{64} \omega(n-3) \quad \text{— ②}$$



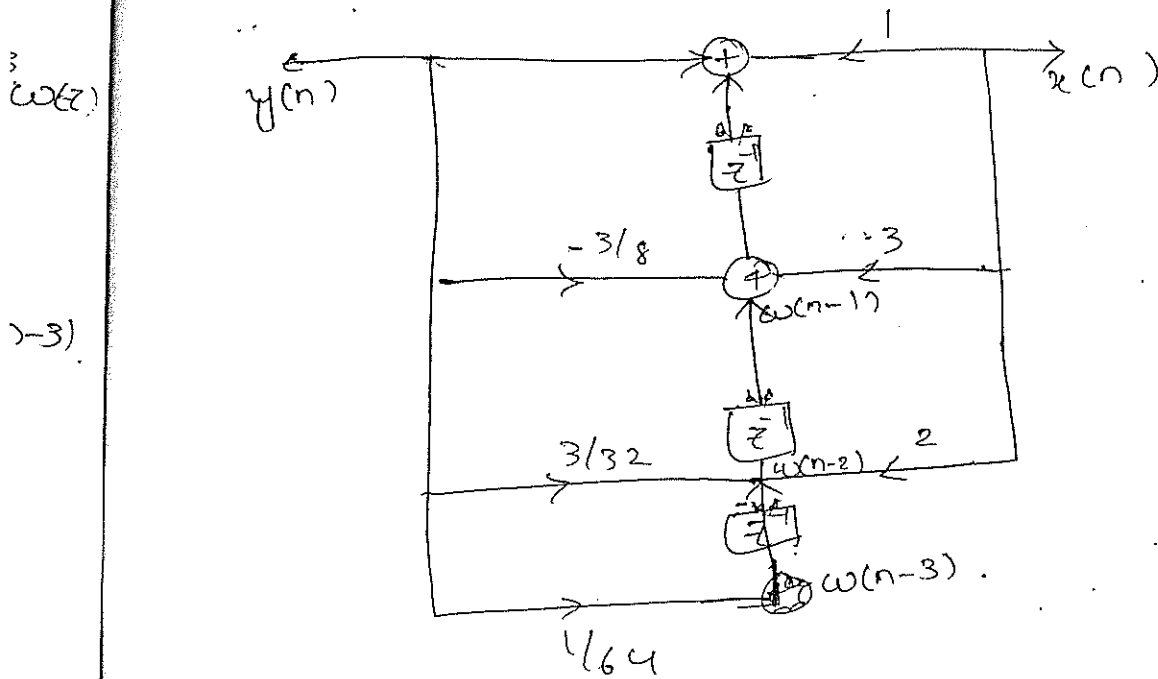
$$\begin{aligned} \text{multipliers} &= M+O+I \\ &= 2+3+1 \\ &= 6 \end{aligned}$$

$$\begin{aligned} \text{adders} &= M+I/O \\ &= 2+3 \\ &= 5 \end{aligned}$$

$$\text{Memory} = \max[2, 3]$$

$$= 3$$

Transposed structure:



multiplicands =

2. Realise the following system using Direct form 1, Direct form 2 & transposed structure.

$$H(z) = \frac{8z^3 - 4z^2 + 11z - 2}{(z - 1/4)(z^2 - z + 1/2)}$$

$$H(z) = \frac{y(z)}{x(z)} = \frac{z^{-3} [8z^3 - 4z^2 + 11z - 2]}{z^{-3} [(z - 1/4)(z^2 - z + 1/2)]}$$

$$= \frac{8 - 4z^{-1} + 11z^{-2} - 2z^{-3}}{1 - z^{-1} + \frac{z^{-2}}{2} - \frac{1}{4}z^{-1} + \frac{z^{-2}}{4} - \frac{1}{8}z^{-3}}$$

$$\frac{y(z)}{x(z)} = \frac{8 - 4z^{-1} + 11z^{-2} - 2z^{-3}}{1 - 5/4z^{-1} + 3/4z^{-2} - 1/8z^{-3}}$$

$$y(z) = \left[1 - \frac{5}{4}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{8}z^{-3} \right] \cdot \left[8 - 4z^{-1} + 11z^{-2} - 2z^{-3} \right] x(z)$$

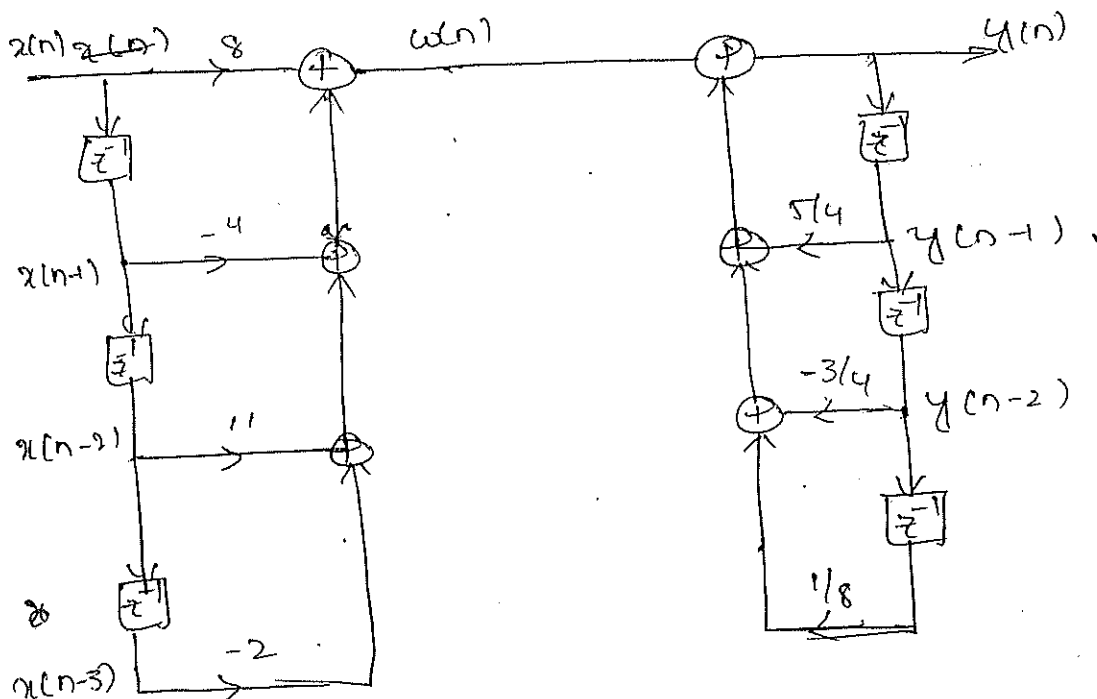
$x(z)$

Apply $z^{-1} \rightarrow T$

$$y(n) = \frac{5}{4}y(n-1) - \frac{3}{4}y(n-2) + \frac{1}{8}y(n-3) + 8x(n) - 4x(n-1) + 11x(n-2) - 2x(n-3)$$

$$w(n) = 8x(n) - 4x(n-1) + 11x(n-2) - 2x(n-3)$$

$$y(n) = \frac{5}{4}y(n-1) - \frac{3}{4}y(n-2) + \frac{1}{8}y(n-3) + w(n)$$



Direct form II

$$\text{adders} = M + N + 1$$

$$= 3 + 3 + 1 = 7$$

$$\text{multipliers} = M + N + 1 = 7$$

$$\text{Memory} = M + N = 6$$

ii)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{8z^0 - 4z^{-1} + 11z^{-2} - 2z^{-3}}{1 - 5/4 z^{-1} + 3/4 z^{-2} - 1/8 z^{-3}}$$

$$= \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}$$

$$\frac{W(z)}{X(z)} = \frac{1}{1 - 5/4 z^{-1} + 3/4 z^{-2} - 1/8 z^{-3}}$$

$$W(z) \left[1 - 5/4 z^{-1} + 3/4 z^{-2} - 1/8 z^{-3} \right] = X(z)$$

Apply I-Z-T

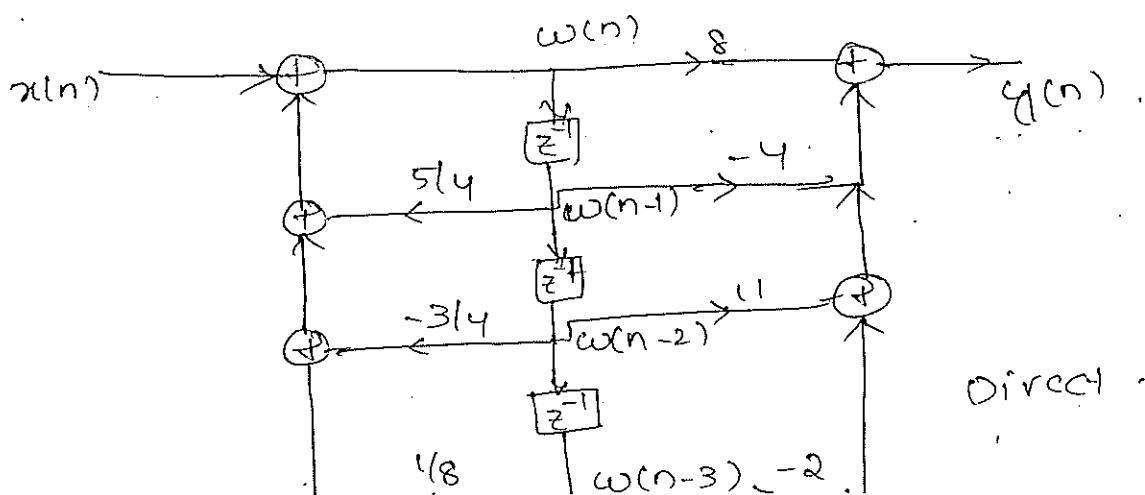
$$W(n) = X(n) + 5/4 W(n-1) - 3/4 W(n-2) + 1/8 W(n-3)$$

$$\frac{Y(z)}{W(z)} = 8 - 4z^{-1} + 11z^{-2} - 2z^{-3}$$

$$Y(z) = (8 - 4z^{-1} + 11z^{-2} - 2z^{-3}) W(z)$$

Apply I-Z-T

$$Y(n) = 8W(n) - 4W(n-1) + 11W(n-2) - 2W(n-3)$$



Direct form II

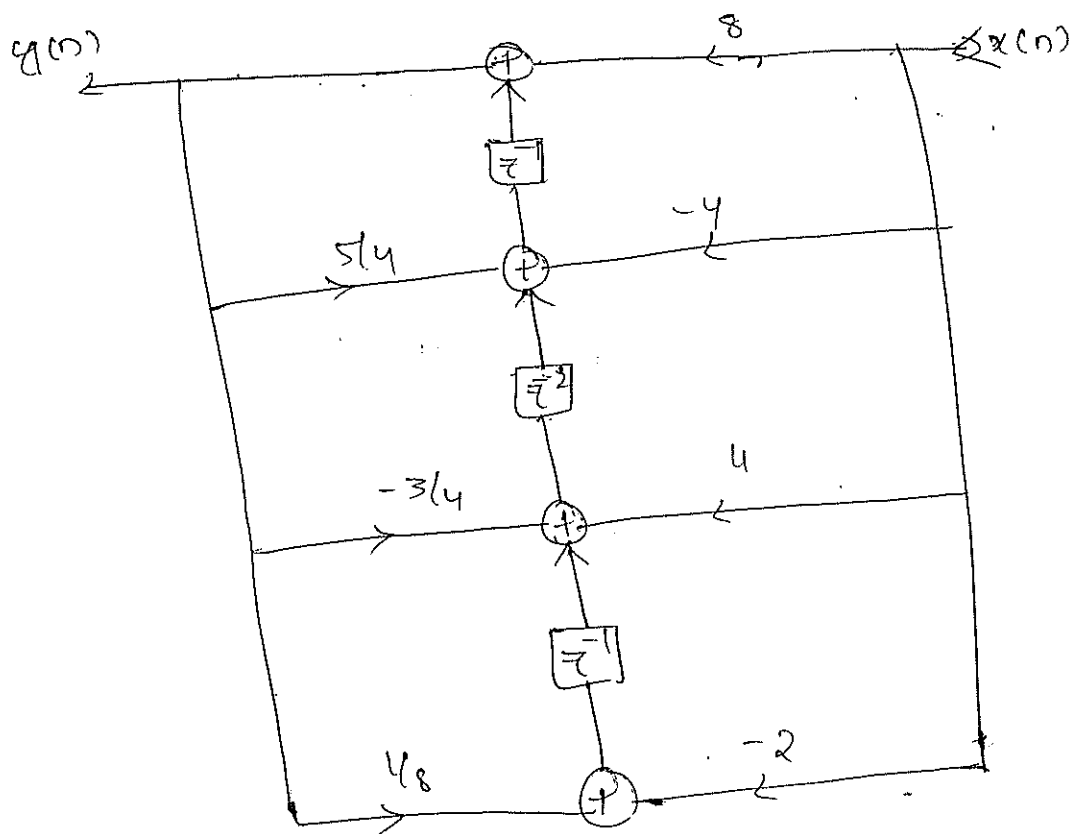
Tr

$$\text{multipliers} = M+10+1 \\ = 3+3+1 \Rightarrow$$

$$\text{Adders} = M+N \\ = 3+3=6.$$

$$\text{Memory} = \max [m, N] \\ = 3.$$

Transposed structure:



* Realize the following system.

$$y(n) = -0.1 y(n-1) + 0.2 y(n-2) + 3x(n) + 3.6 x(n-1) + 0.6$$

$x(n-2)$. by using i) Direct form ii) Direct form iii) cascade Transposed iv) cascade v) parallel realizations.

Cascade Realization:

$$y(n) = -0.1 y(n-1) + 0.2 y(n-2) + 3x(n) + 3.6 x(n-1) + 0.6 x(n-2)$$

Applying z-transform on both sides

$$Y(z) = -0.1 z^{-1} Y(z) + 0.2 z^{-2} Y(z) + 3 X(z) + 3.6 z^{-1} X(z) + 0.6 z^{-2} X(z)$$

$$Y(z) [1 + 0.1 z^{-1} - 0.2 z^{-2}] = [3 + 3.6 z^{-1} + 0.6 z^{-2}] X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{3 + 3.6 z^{-1} + 0.6 z^{-2}}{1 + 0.1 z^{-1} - 0.2 z^{-2}} \times \frac{z^2}{z^2}$$

$$= \frac{3z^2 + 3.6z + 0.6}{z^2 + 0.1z - 0.2}$$

$$= \frac{(3z + 0.6)(z + 1)}{(z + 0.5)(z - 0.4)}$$

$$H(z) = \left[\frac{3z + 0.6}{z + 0.5} \right] \left[\frac{z + 1}{z - 0.4} \right] = \left[\frac{3 + 0.6 z^{-1}}{1 + 0.5 z^{-1}} \right] \left[\frac{1 + z^{-1}}{1 - 0.4 z^{-1}} \right]$$

$$= H_1(z) \cdot H_2(z)$$

$$H_K(z) = \frac{3 + 0.6 z^{-1}}{1 + 0.5 z^{-1}} = \frac{A_K + b_K z^{-1}}{1 + a_K z^{-1}} = \frac{Y_K(z)}{X_K(z)} = \frac{Y_K(z)}{\omega_K(z)} \cdot \frac{\omega_K(z)}{X_K(z)}$$

$$\frac{\omega_K(z)}{X_K(z)} = \frac{1}{1 + a_K z^{-1}} \Rightarrow \omega_K(z) + \omega_K(z) a_K z^{-1} = X_K(z)$$

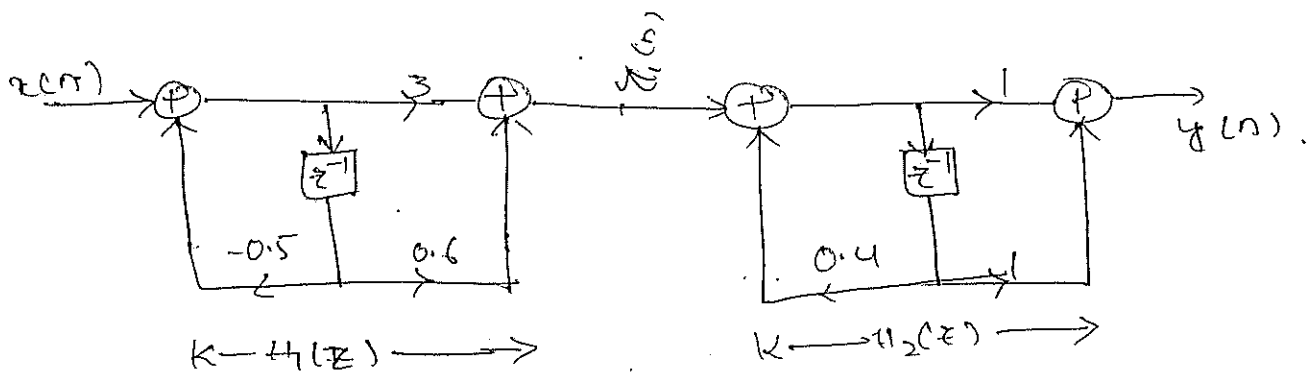
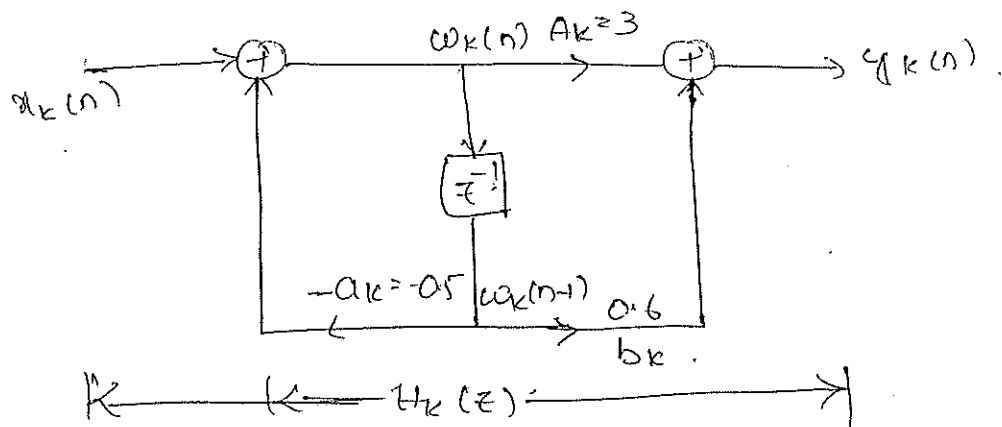
$$\omega_K(z) = X_K(z) - a_K z^{-1} \omega_K(z)$$

$$\omega_K(n) = X_K(n) - a_K \omega_K(n-1)$$

$$\frac{Y_K(z)}{\omega_K(z)} = A_K + b_K z^{-1} \Rightarrow Y_K(z) = A_K \omega_K(z) + b_K z^{-1} \omega_K(z)$$

$$Y_K(n) = A_K \omega_K(n) + b_K \omega_K(n-1)$$

← (2)



$$\leftarrow H(z) = H_1(z) \cdot H_2(z) \rightarrow$$

Parallel form Realization:

$$\begin{aligned}
 H(z) &= \frac{Y(z)}{X(z)} = \frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}} \\
 &= \frac{3z^2 + 3.6z + 0.6}{z^2 + 0.1z - 0.2} = \frac{(3z + 0.6)(z + 1)}{(z + 0.5)(z - 0.4)} \\
 &\overset{X}{=} 3 + \frac{3 \cdot 3z + 1.2}{z^2 + 0.1z - 0.2} \quad \begin{array}{l} \text{3} \\ \hline 3z^2 + 3.6z + 0.6 \\ 3z^2 + 0.3z - 0.6 \\ \hline (-) \quad (-) \quad (+) \\ 3.3z + 1.2 \end{array} \\
 &= 3 + \frac{3.3z + 1.2}{(z + 0.5)(z - 0.4)} \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \frac{H(z)}{z} &= \frac{3z + 0.6}{z} \cdot \frac{3z^2 + 3.6z + 0.6}{z(z^2 + 0.1z - 0.2)} = \frac{3z^2 + 3.6z + 0.6}{z(z + 0.5)(z - 0.4)} \\
 &= \frac{A}{z} + \frac{B}{z + 0.5} + \frac{C}{z - 0.4}
 \end{aligned}$$

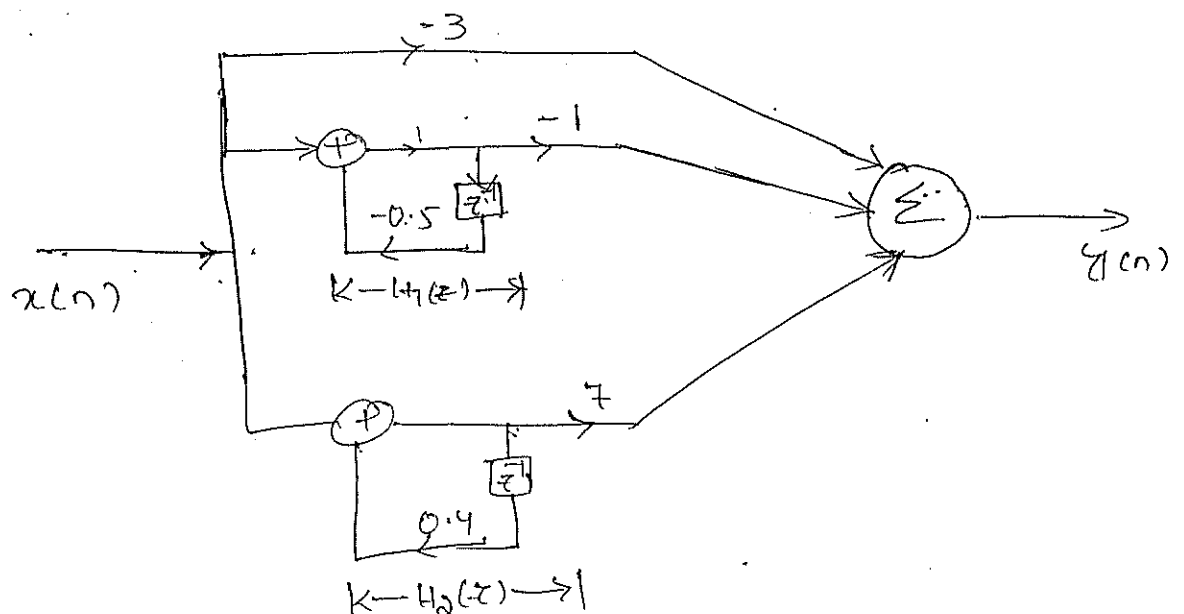
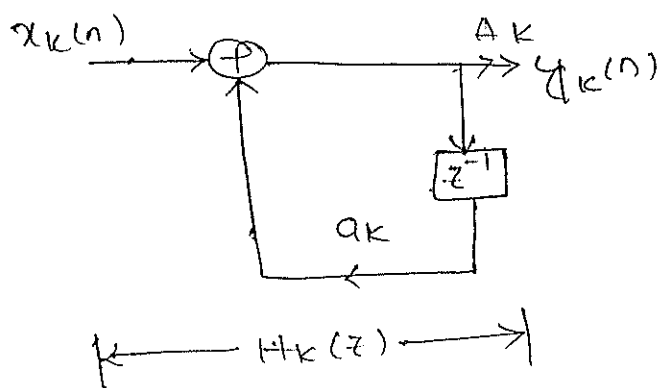
$$\frac{H(z)}{z} = -\frac{3}{z} + \frac{-1}{z + 0.5} + \frac{4}{z - 0.4}$$

$$H(z) = -3 + \frac{-z}{z+0.5} + \frac{7z}{z-0.4}$$

$$H(z) = -3 + \frac{-1}{1+0.5z^{-1}} + \frac{7}{1-0.4z^{-1}}$$

$$= A + H_1(z) + H_2(z)$$

$$H_k(z) = \frac{A_k}{1-a_k z^{-1}} = \frac{y_k(z)}{x_k(z)}$$



FIR Digital Filters structures (or) Non Recursive Realization techniques:

There are 3 well known techniques to realize FIR digital filters.

1. direct form realization (or) Transversal realization
2. -on (or) Tapped delay line filter.

2. cascade Realization

3. Linear phase Realization (or) Min no. of multiplier-realization technique.

1. Direct form Realization technique:

If the system having finite duration impulse response seqs sequence then it is called FIR system.

If the length of impulse response is

N

i.e.,

$$h(n) = \{h(0), h(1), h(2), \dots, h(N-1)\};$$

length

$$T.F = H(z) = \frac{Y(z)}{X(z)} = Z.T[h(n)] = \sum_{n=0}^{N-1} [h(n)] z^{-n} \\ = \sum_{k=0}^{N-1} [h(k)] z^{-k}$$

$$Y(z) = \sum_{k=0}^{N-1} h(k) z^{-k} X(z)$$

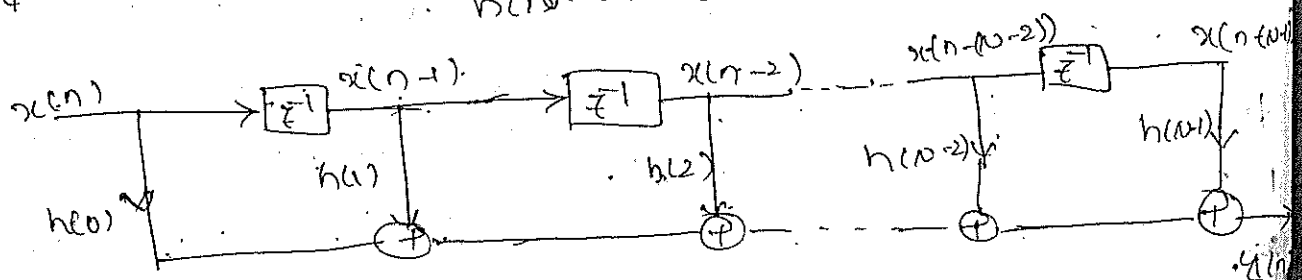
Applying I.Z.T & linearity property, we get

$$I.Z.T[Y(z)] = I.Z.T \left[\sum_{k=0}^{N-1} h(k) z^{-k} X(z) \right]$$

$$y(n) = \sum_{k=0}^{N-1} h(k) I.Z.T[z^{-k} X(z)]$$

$$= \sum_{k=0}^{N-1} h(k) x(n-k)$$

$$y(n) = h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \dots + h(N-1)x(n-(N-1)) \quad \text{--- (1)}$$



This realization requires,

- total no. of multipliers = N .
- total no. of adders = $N-1$.
- Memory locations = $N-1$.

Cascade Realization:

This realization is implemented all the subsystems of system function $H(z)$ can be connected in series i.e., cascade.

Case (i): If ' N ' is odd

$$H(z) = \prod_{k=1}^{\frac{N-1}{2}} H_k(z) = \prod_{k=1}^{\frac{N-1}{2}} [b_{0k} + b_{1k}z^{-1} + b_{2k}z^{-2}]$$

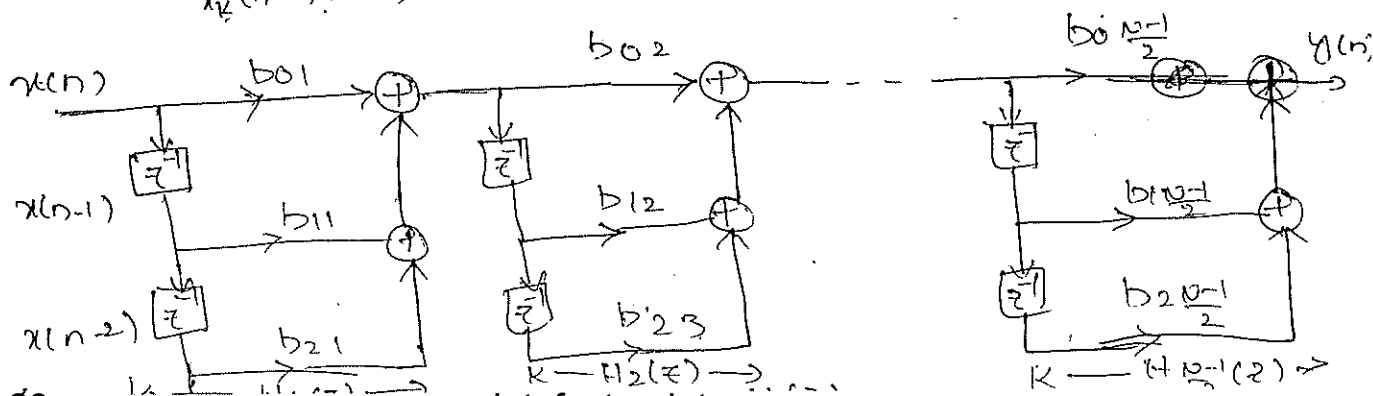
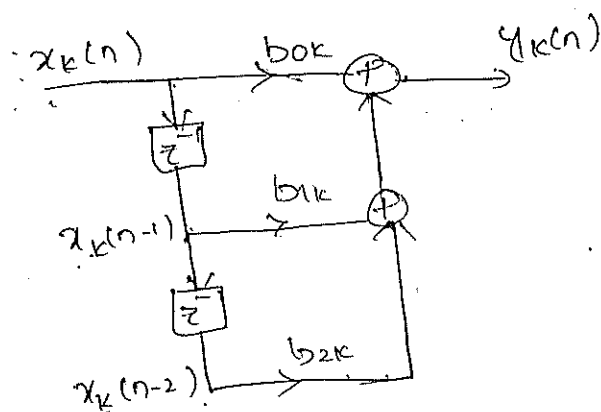
$$= H_1(z) \cdot H_2(z) \cdot \dots \cdot H_{\frac{N-1}{2}}(z)$$

$N = \text{odd}$, $\frac{N-1}{2} = \text{even}$ $\therefore H(z)$ having $\left(\frac{N-1}{2}\right)$

second order systems.

$$H_k(z) = \frac{y_k(z)}{x_k(z)} = b_{0k} + b_{1k}z^{-1} + b_{2k}z^{-2}$$

$$y_k(n) = b_{0k} x_k(n) + b_{1k} x_k(n-1) + b_{2k} x_k(n-2) \quad \text{--- (1)}$$



If N is even, it has one first order system and $\frac{N}{2}-1$ second order systems.

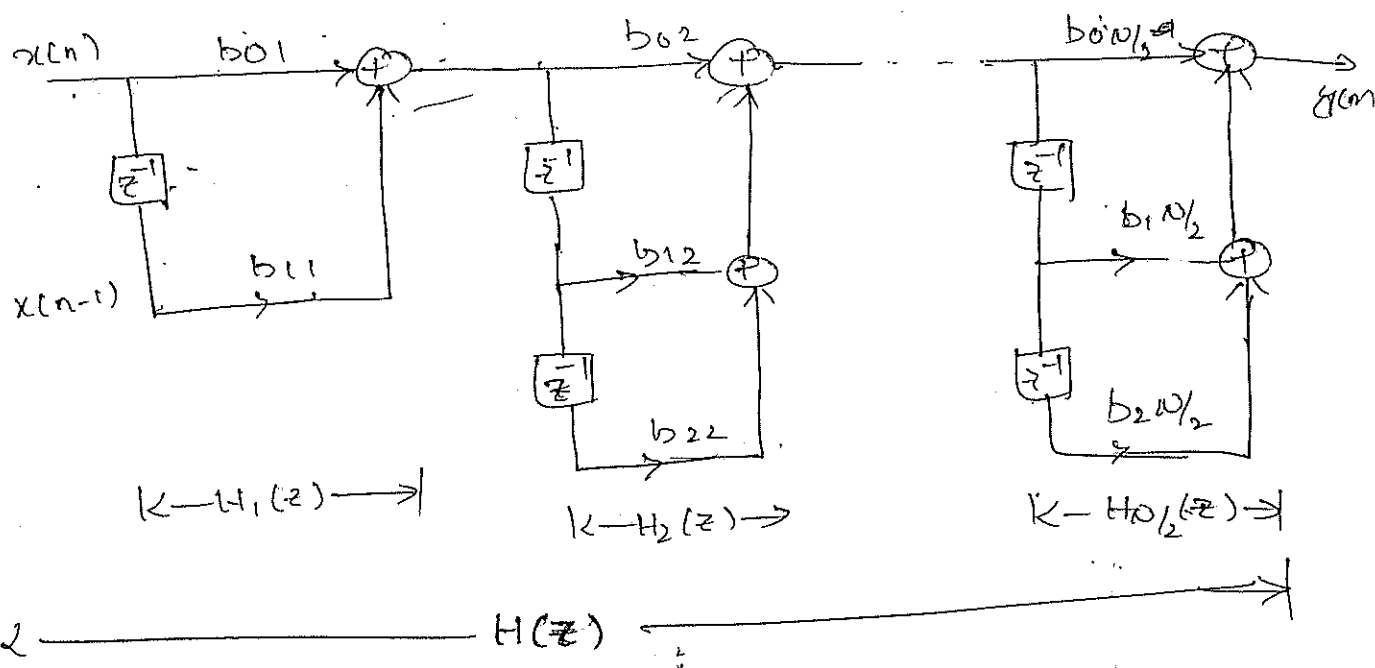
$$H(z) = (b_0 + b_1 z^{-1}) \prod_{k=2}^{N/2} (1 + p_k(z))$$

$$= (b_0 + b_1 z^{-1}) \prod_{k=2}^{N/2} [b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2}]$$

$$= (b_0 + b_1 z^{-1}) H_2(z) \cdot H_3(z) \cdots H_{N/2}(z)$$

$$H_k(z) = \frac{y_k(z)}{x_k(z)} = b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2}$$

$$y_k(n) = b_{0k} x_k(n) + b_{1k} x_k(n-1) + b_{2k} x_k(n-2)$$



* Realize the following system using direct form realization.

$$h(n) = \delta(n) - 2\delta(n-1) + 3\delta(n-2) + 2\delta(n-4) + 5\delta(n-6) - \delta(n-7)$$

Sol:-

$$h(n) = \{1, -2, 3, 0, 2, 0, 5, -1\}$$

0 1 2 3 4 5 6 7

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

$$= \sum_{n=0}^7 h(n)z^{-n}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} +$$

$$h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7}$$

$$H(z) = \frac{Y(z)}{X(z)} = 1 - 2z^{-1} + 3z^{-2} + 2z^{-4} + 5z^{-6} - z^{-7}$$

$$Y(z) = X(z) - 2z^{-1}X(z) + 3z^{-2}X(z) + 2z^{-4}X(z) +$$

$$5z^{-6}X(z) - z^{-7}X(z)$$

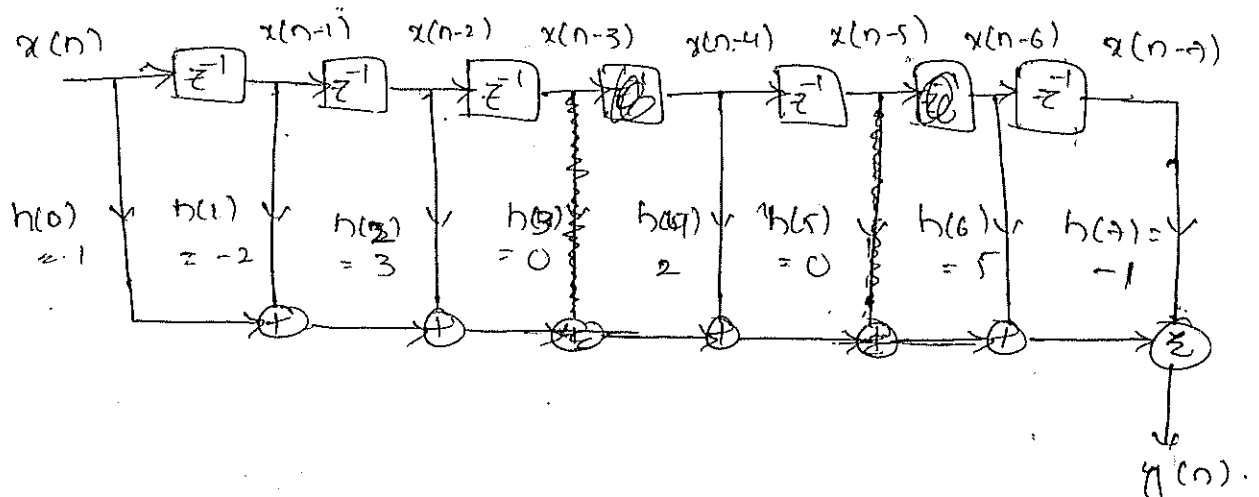
$$5z^{-6}X(z) - z^{-7}X(z)$$

Apply 1.z.T & linearity property of z.T

we get

$$Y(n) = X(n) - 2X(n-1) + 3X(n-2) + 2X(n-4) + 5$$

$$X(n-6) - X(n-7) \text{ --- (1)}$$



* Realize the following system using

1. Cascade Realization
2. Direct form Realization.

$$H(z) = (1 - z^{-1} + 2z^{-2} + 3z^{-3} - z^{-4})(2 - z^{-1} - 4z^{-3} + 5z^{-5})$$

$$H(z) = H_1(z) \cdot H_2(z)$$

$$H_1(z) = 1 - z^{-1} + 2z^{-2} + 3z^{-3} - z^{-4} = \frac{Y_1(z)}{X_1(z)}$$

$$H_2(z) = 2 - z^{-1} - 4z^{-3} + 5z^{-5} = \frac{Y_2(z)}{X_2(z)}$$

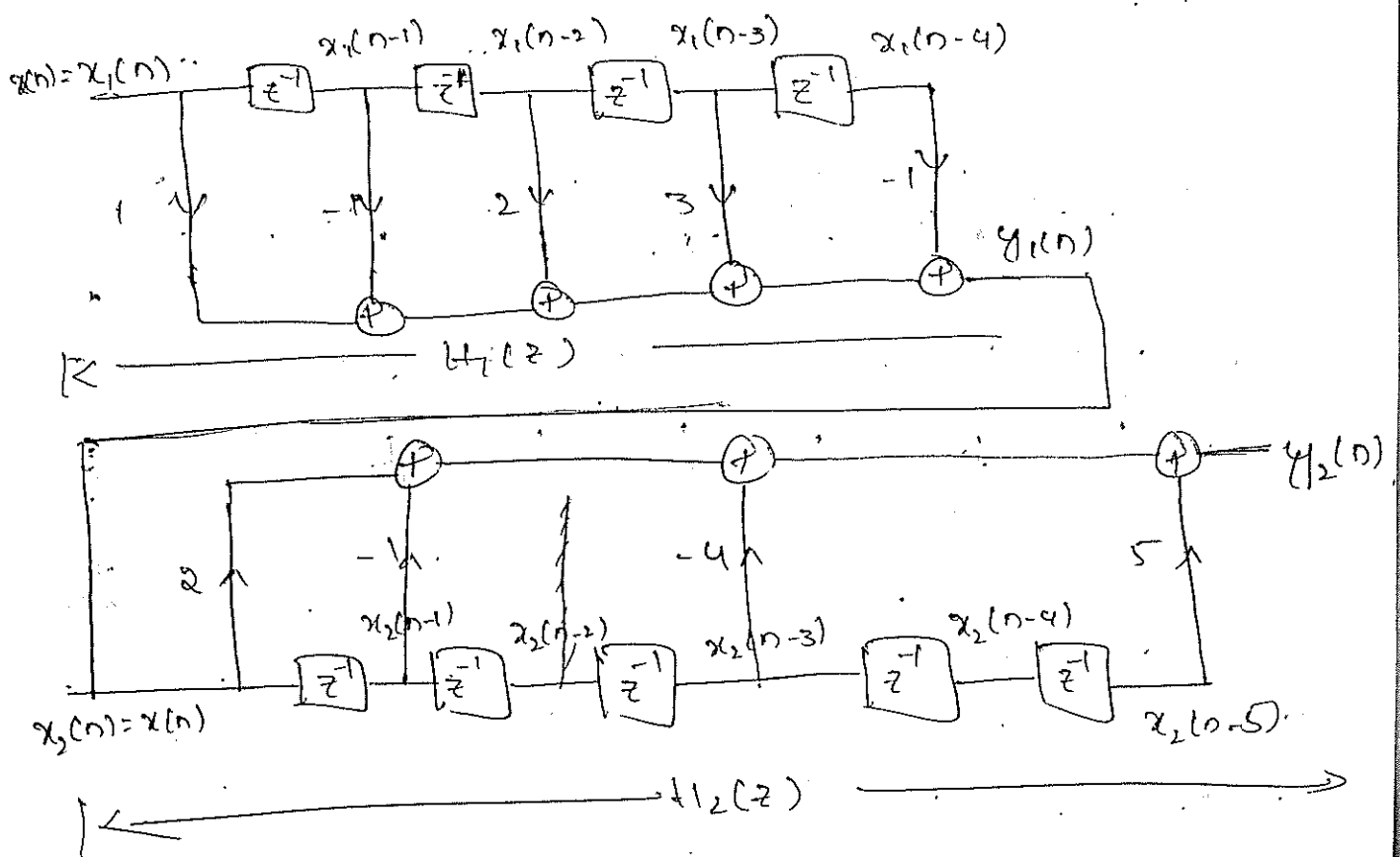
$$y_1(n) =$$

$$y_1(z) = x_1(z) - z^{-1}x_1(z) + 2z^{-2}x_1(z) + 3z^{-3}x_1(z) - x_1(z)z^{-4}$$

$$y_1(n) = x_1(n) - x_1(n-1) + 2x_1(n-2) + 3x_1(n-3) - x_1(n-4) \quad \text{--- (1)}$$

$$y_2(n) = 2x_2(n) - x_2(n)z^{-1} - 4z^{-3}x_2(z) + 5z^{-5}x_2(z)$$

$$y_2(n) = 2x_2(n) - x_2(n-1) - 4x_2(n-3) + 5x_2(n-5) \quad \text{--- (2)}$$



Direct form:

$$H(z) = (1 - z^{-1} + 2z^{-2} + 3z^{-3} - z^{-4})(2 - z^{-1} - 4z^{-3} + 5z^{-5})$$

$$= 2 - z^{-1} - 4z^{-3} + 5z^{-5} - 2z^{-1} + z^{-2} + 4z^{-4} - 5z^{-6}$$

$$+ 4z^{-2} - 2z^{-3} - 8z^{-5} + 10z^{-7} + 6z^{-3} - 3z^{-4}$$

$$+ 12z^{-6} + 15z^{-8} - 2z^{-4} + z^{-5} + 4z^{-7} - 5z^{-9}$$

$$z = 2 - 3z^{-1} + 5z^{-2} + z^{-4}$$

$x_1(z)$

$1-3)$

$x_2(z)$

$1-5)$

$10)$

$z^{-5})$

-6

Linear phase Realization technique / Minimum no. of multipliers required realization:-

for linear phase FIR digital filter, impulse response satisfies the following condition.

$$h(n) = h(N-1-n) \quad \forall 'n'$$

N - length of $h(n)$

$$h(n) = \{h(0), h(1), h(2), \dots, h(N-1)\}$$

$$\begin{aligned} T.F = z.T[h(n)] &= H(z) = \frac{Y(z)}{X(z)} = \sum_{n=0}^{N-1} h(n) z^{-n} \\ &= h(0) + h(1)z^{-1} + h(2)z^{-2} + \dots \\ &\quad \dots + h(N-1)z^{-(N-1)} \end{aligned}$$

Case(i) :-

If ' N ' is even

$$\begin{aligned} T.F = z.T[h(n)] &= H(z) = \frac{Y(z)}{X(z)} = \sum_{n=0}^{N-1} h(n) z^{-n} \\ &= \sum_{n=0}^{N-1} [h(n)] z^{-n} + \sum_{n=\frac{N}{2}}^{N-1} [h(n)] z^{-n} \end{aligned}$$

$$\text{put } n = N-1-m$$

$$m = N-1-n$$

$$n = N/2; \quad m \rightarrow (N-1) - \frac{N}{2} = \frac{N}{2} - 1$$

$$n = N-1; \quad m \rightarrow (N-1) - (N-1) = 0$$

$$z^{-n} + \sum_{m=\frac{N}{2}-1}^0 h(N-1-m) z^{-(N-1-m)}$$

$$\text{put } m = n$$

$$= \sum_{n=0}^{N/2-1} h(n) z^{-n} + \sum_{n=0}^{N/2-1} h(N-1-n) z^{-(N-1-n)}$$

$$\therefore h(n) = h(N-1-n)$$

o/p

$$= \sum_{n=0}^{N/2-1} h(n)z^{-n} + \sum_{n=0}^{N/2-1} h(n)z^{-(N-1-n)}$$

i/p

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{n=0}^{N/2-1} h(n) \left[z^{-n} + z^{-(N-1-n)} \right]$$

$$Y(z) = \sum_{n=0}^{N/2-1} h(n) \left[z^{-n} X(z) + z^{-(N-1-n)} X(z) \right]$$

put $n=k$

$$Y(z) = \sum_{k=0}^{N/2-1} h(k) \left[z^{-k} X(z) + z^{-(N-1-k)} X(z) \right]$$

Applying 1. z.T. & linearity property.

$$y(k) = \sum_{k=0}^{N/2-1} h(k) \left[x(n-k) + x(n-(N-1-k)) \right]$$

$$y(n) = \sum_{k=0}^{N/2-1} h(k) \left[x(n-k) + x(n-(N-1-k)) \right]$$

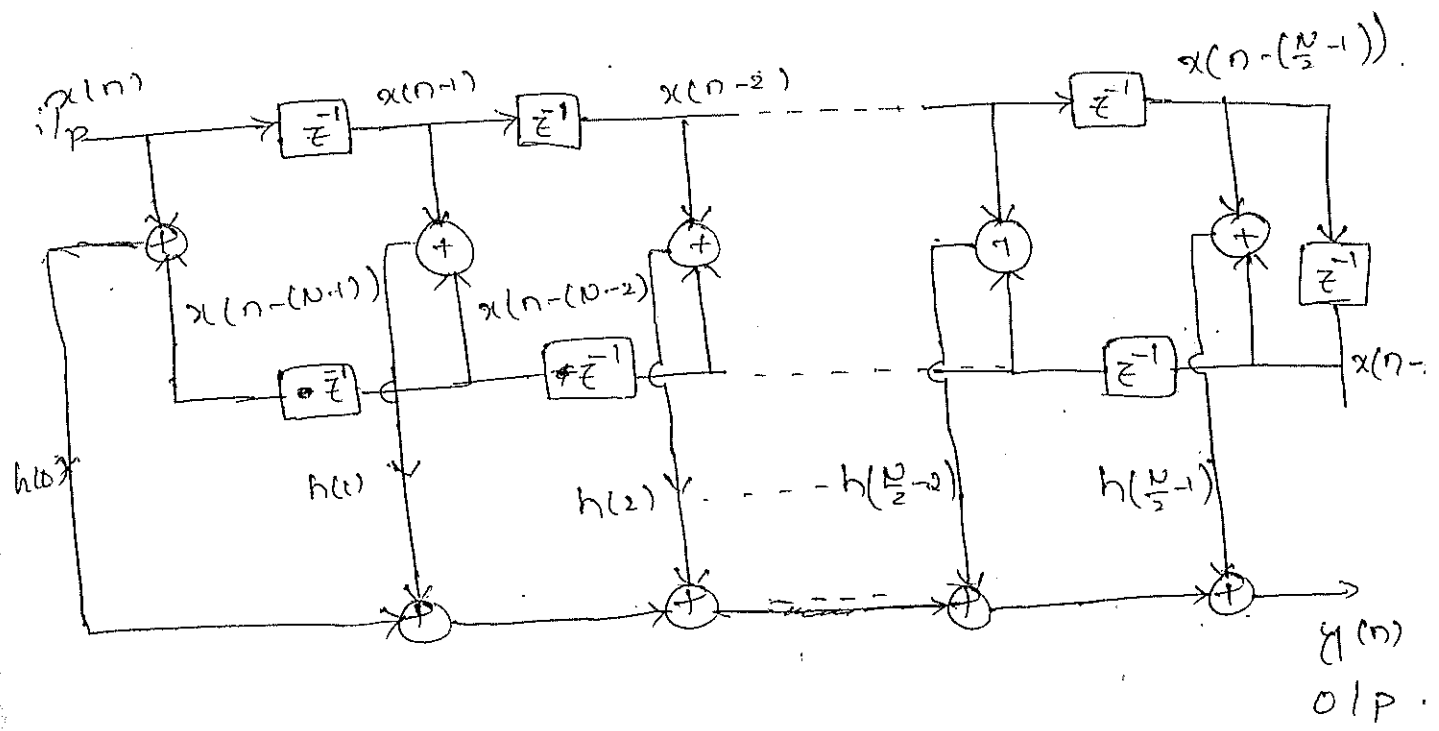
z^{-n}

$$y(n) = h(0) [x(n) + x(n-(N-1))] + h(1) [x(n-1) + x(n-(N-2))] +$$

$$+ h\left(\frac{N}{2}-1\right) [x(n-(\frac{N}{2}-1)) + x(n-(\frac{N}{2}))] \quad \text{--- ①}$$

o/p
o/p

o/p



for Direct form realization, total no. of multipliers required = N .

for linear phase realization & N is even,
total no. of multipliers required = $N/2$.

Case (ii) :-

If N is odd number.

$$\begin{aligned} T.F = z \cdot T[h(n)] &= H(z) = \frac{Y(z)}{X(z)} = \sum_{n=0}^{N-1} h(n) z^{-n} \\ &= \sum_{n=0}^{\frac{N-3}{2}} h(n) z^{-n} + h\left(\frac{N-1}{2}\right) z^{-\left(\frac{N-1}{2}\right)} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n) z^{-n} \end{aligned}$$

$$\text{put } n = N-1-m$$

$$m = N-1-n$$

$$n \rightarrow \frac{N+1}{2}; \therefore m \rightarrow (N-1) - \left(\frac{N+1}{2}\right) = \frac{N-3}{2}$$

$$n \rightarrow N-1; m = 0$$

$$= h\left(\frac{N-1}{2}\right) z^{-\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(n) z^{-n} + \sum_{m=\frac{N-3}{2}}^0 h(N-1-m) z^{-(N-1-m)}$$

$$\text{put } m = n$$

$$= h\left(\frac{N-1}{2}\right) z^{-\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(n) z^{-n} + \sum_{n=0}^{\frac{N-3}{2}} h(N-1-n) z^{-(N-1-n)}$$

$$\therefore h(n) = h(N-1-n)$$

$$= h\left(\frac{N-1}{2}\right) z^{-\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(n) z^{-n} + \sum_{n=0}^{\frac{N-3}{2}} h(n) z^{-(N-1-n)}$$

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{\infty}$$

$$= h\left(\frac{N-1}{2}\right) z^{-\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(n) \left[z^{-n} + z^{-(N-1-n)} \right]$$

Let

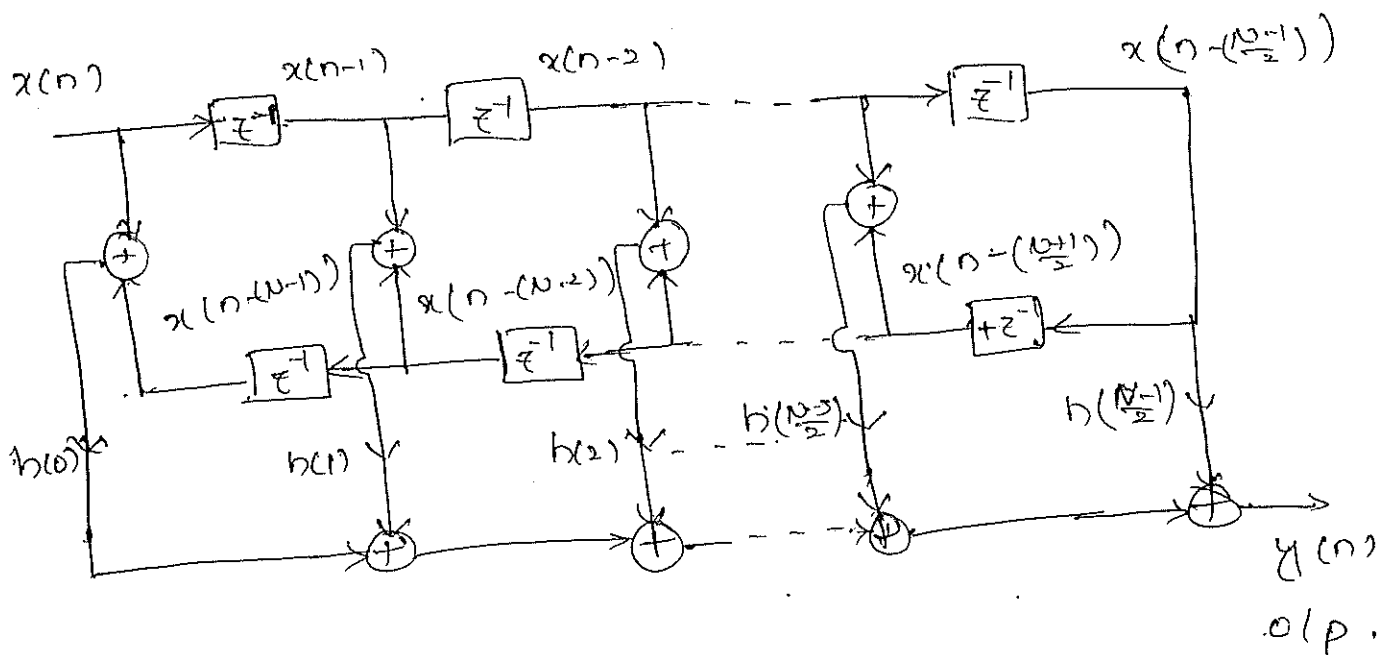
$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{\frac{N-3}{2}} h(k) \left[z^{-k} + z^{-(N-1-k)} \right] + h\left(\frac{N-1}{2}\right) z^{-\left(\frac{N-1}{2}\right)}$$

$$Y(z) = \sum_{k=0}^{\frac{N-3}{2}} h(k) \left[z^{-k} X(z) + z^{-(N-1-k)} X(z) \right] + h\left(\frac{N-1}{2}\right) z^{-\left(\frac{N-1}{2}\right)} X(z)$$

Applying L.T. & linearity prop.

$$y(n) = \sum_{k=0}^{\frac{N-3}{2}} h(k) \left[x(n-k) + x(n-(N-1-k)) \right] + h\left(\frac{N-1}{2}\right) x\left(n - \left(\frac{N-1}{2}\right)\right)$$

$$y(n) = h(0) \left[x(n) + x(n-(N-1)) \right] + h(1) \left[x(n-1) + x(n-(N-2)) \right] + \dots + h\left(\frac{N-3}{2}\right) \left[x\left(n - \left(\frac{N-3}{2}\right)\right) + x\left(n - \left(\frac{N+1}{2}\right)\right) \right] + h\left(\frac{N-1}{2}\right) x\left(n - \left(\frac{N-1}{2}\right)\right) \quad \text{--- (1)}$$



for $N = \text{odd}$.

$$\text{no. of multipliers} = \frac{N+1}{2}$$

* Realise the following system using...

i) Direct form ii) linear phase iii) ~~cascade~~ R.T.

$$h(n) = \frac{1}{2} \delta(n) + \frac{1}{3} \delta(n-1) + \delta(n-2) + \frac{1}{4} \delta(n-3) + \delta(n-4) \\ + \frac{1}{3} \delta(n-5) + \frac{1}{2} \delta(n-6).$$

Sol:- i) Direct form:-

$$h(n) = \left\{ \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{4}, 1, \frac{1}{3}, \frac{1}{2} \right\}$$

length of $h(n) = N = 7$.

$$H(z) = z^{-1} T [h(n)] = \sum_{n=0}^{N-1} h(n) z^{-n}$$

$$= \sum_{n=0}^6 h(n) z^{-n}$$

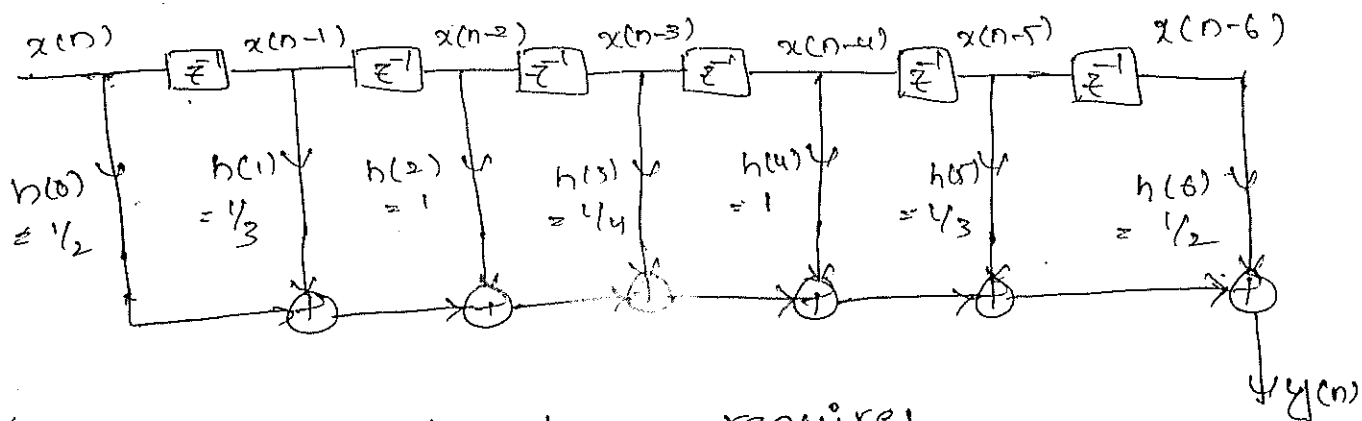
$$= h(0) + h(1) z^{-1} + h(2) z^{-2} + h(3) z^{-3} + h(4) z^{-4} \\ + h(5) z^{-5} + h(6) z^{-6}.$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{2} + \frac{1}{3} z^{-1} + z^{-2} + \frac{1}{4} z^{-3} + z^{-4} + \frac{1}{3} z^{-5} + \frac{1}{2} z^{-6}$$

$$Y(z) = \frac{1}{2} X(z) + \frac{1}{3} z^{-1} X(z) + z^{-2} X(z) + \frac{1}{4} z^{-3} X(z) + z^{-4} X(z) \\ + \frac{1}{3} z^{-5} X(z) + \frac{1}{2} z^{-6} X(z).$$

Applying I.Z.T & linearity prop.

$$y(n) = \frac{1}{2} x(n) + \frac{1}{3} x(n-1) + x(n-2) + \frac{1}{4} x(n-3) \\ + x(n-4) + \frac{1}{3} x(n-5) + \frac{1}{2} x(n-6) \quad \text{--- (1)}$$



This realization requires,

no. of multipliers = 7.

no. of adders = 7 - 1 = 6.

no. of memory locations = 7 - 1 = 6.

ii) Linear phase:

4) z^{-4}
we know condition for linear phase filter,

$$h(n) = h(N-1-n), \forall n.$$

$$N=7; \quad h(n) = h(6-n).$$

$$n=0; \quad h(0) = h(6) = 1/2.$$

$$n=1; \quad h(1) = h(5) = 1/3.$$

$$n=2; \quad h(2) = h(4) = 1.$$

$$n=3; \quad h(3) = 1/4 = h(3).$$

∴ Given system satisfies linear phase condition.

$$H(z) = z^{-4} [h(n)] = \sum_{n=0}^6 h(n) z^{-n}.$$

$$H(z) = \frac{Y(z)}{X(z)} = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6}.$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(2)z^{-4} + h(1)z^{-5} + h(0)z^{-6}$$

$$h(0)[1+z^{-6}]$$

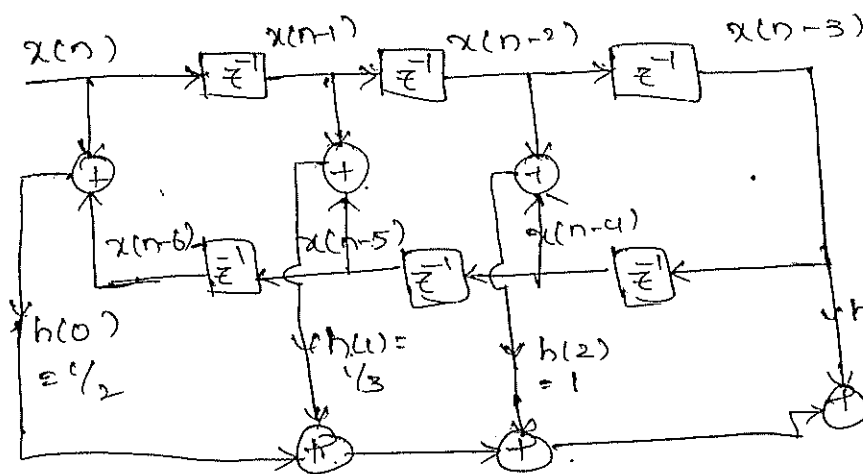
$$= h(0) + h(1)[z^{-1} + z^{-5}] + h(2)[z^{-2} + z^{-4}] + h(3)z^{-3}$$

$$y(z) = h(0)[x(z) + z^{-6}x(z)] + h(1)[z^{-1}x(z) + z^{-5}x(z)] + h(2)[z^{-2}x(z) + z^{-4}x(z)] + h(3)z^{-3}x(z)$$

Applying 1. z^{-1} & linearity prop.

$$y(n) = h(0)[x(n) + x(n-6)] + h(1)[x(n-1) + x(n-5)] + h(2)[x(n-2) + x(n-4)] + h(3)x(n-3)$$

①



no. of multipliers

$$= \frac{N+1}{2} = 4$$

$$\text{adders} = \frac{N-1}{2}$$

$$= 3$$

$$\text{memory} = 6$$

$$2. h(n) = \frac{1}{2} \delta(n) + \frac{1}{3} \delta(n-1) + \delta(n-2) + \frac{1}{4} \delta(n-3) + \frac{1}{4} \delta(n-4) + \delta(n-5) + \frac{1}{3} \delta(n-6) + \frac{1}{2} \delta(n-7)$$

Sol:- Linear phase:

$$h(n) = \left\{ \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{4}, \frac{1}{4}, 1, \frac{1}{3}, \frac{1}{2} \right\}$$

$$N = 8$$

Condition for linear phase.

$$h(n) = h(N-1-n) \quad \forall n$$

$$n=0; \quad h(4-0) = h(0) = h(7) = \frac{1}{2}$$

$$n=1; \quad h(1) = h(6) = \frac{1}{3}$$

$$n=2; \quad h(2) = h(5) = 1$$

$$n=3; \quad h(3) = h(4) = \frac{1}{4}$$

$\therefore$ Given system satisfies linear phase condition.

$$H(z) = z^{-T} [h(n)] = \sum_{n=0}^{N-1} h(n) z^{-n}$$

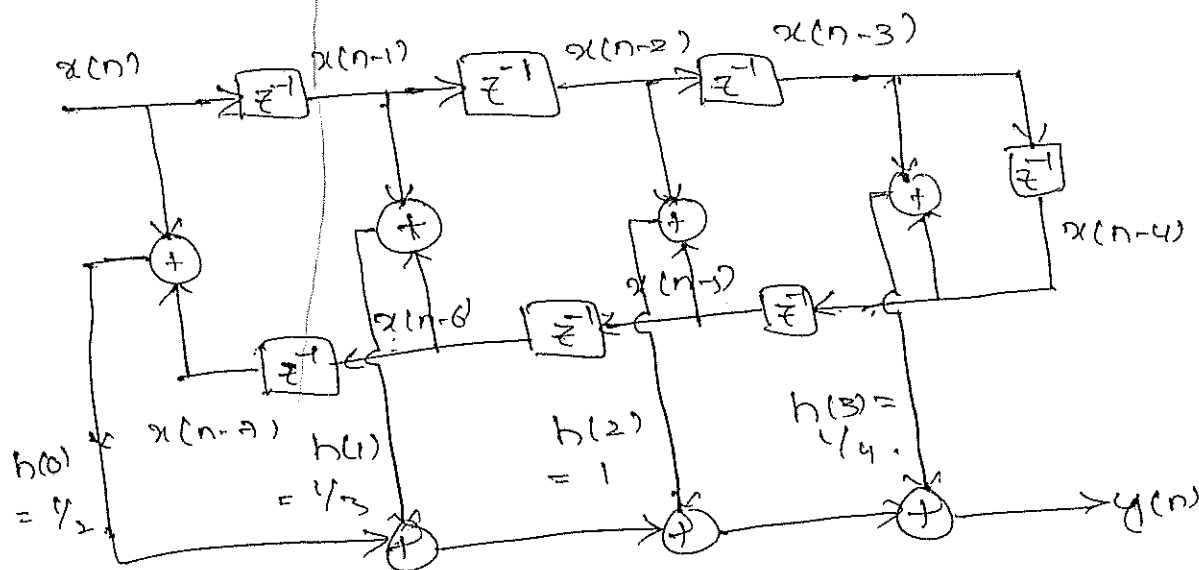
$$\begin{aligned} H(z) = \frac{y(z)}{x(z)} &= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7} \\ &= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7} \\ &= h(0)[1 + z^{-7}] + h(1)[z^{-1} + z^{-6}] + h(2)[z^{-2} + z^{-5}] + h(3)[z^{-3} + z^{-4}] \end{aligned}$$

$$\begin{aligned} y(z) &= h(0)[x(z) + z^{-7}x(z)] + h(1)[z^{-1}x(z) + z^{-6}x(z)] \\ &+ h(2)[z^{-2}x(z) + z^{-5}x(z)] + h(3)[z^{-3}x(z) + z^{-4}x(z)] \end{aligned}$$

Applying I.Z.T and linearity prop

$$y(n) = h(0) [x(n) + x(n-4)] + h(1) [x(n-1) + x(n-3)] + h(2) [x(n-2) + x(n-2)] + h(3) [x(n-3) + x(n-1)]$$

②



Total no. of multipliers used = $4 \cdot \frac{1}{2} \cdot \frac{1}{2} = 8 \cdot \frac{1}{2} = 4$.

adders = $N-1 = 3$.

memory elements = 7.

Cascade -

$$3. H(z) = \left(1 + \frac{1}{2}z^{-1} + z^{-2}\right) \left(\frac{1}{2} + z^{-1} + \frac{1}{2}z^{-2}\right) \\ = H_1(z) H_2(z)$$

$$H_1(z) = \frac{y_1(z)}{x_1(z)} = 1 + \frac{1}{2}z^{-1} + z^{-2}$$

$$y_1(z) = x_1(z) + \frac{1}{2}z^{-1}x_1(z) + x_1(z)z^{-2}$$

$$y_1(n) = x_1(n) + \frac{1}{2}x_1(n-1) + x_1(n-2)$$

$$= [x_1(n) + x_1(n-2)] + \frac{1}{2}x_1(n-1) \text{ --- (1)}$$

key .

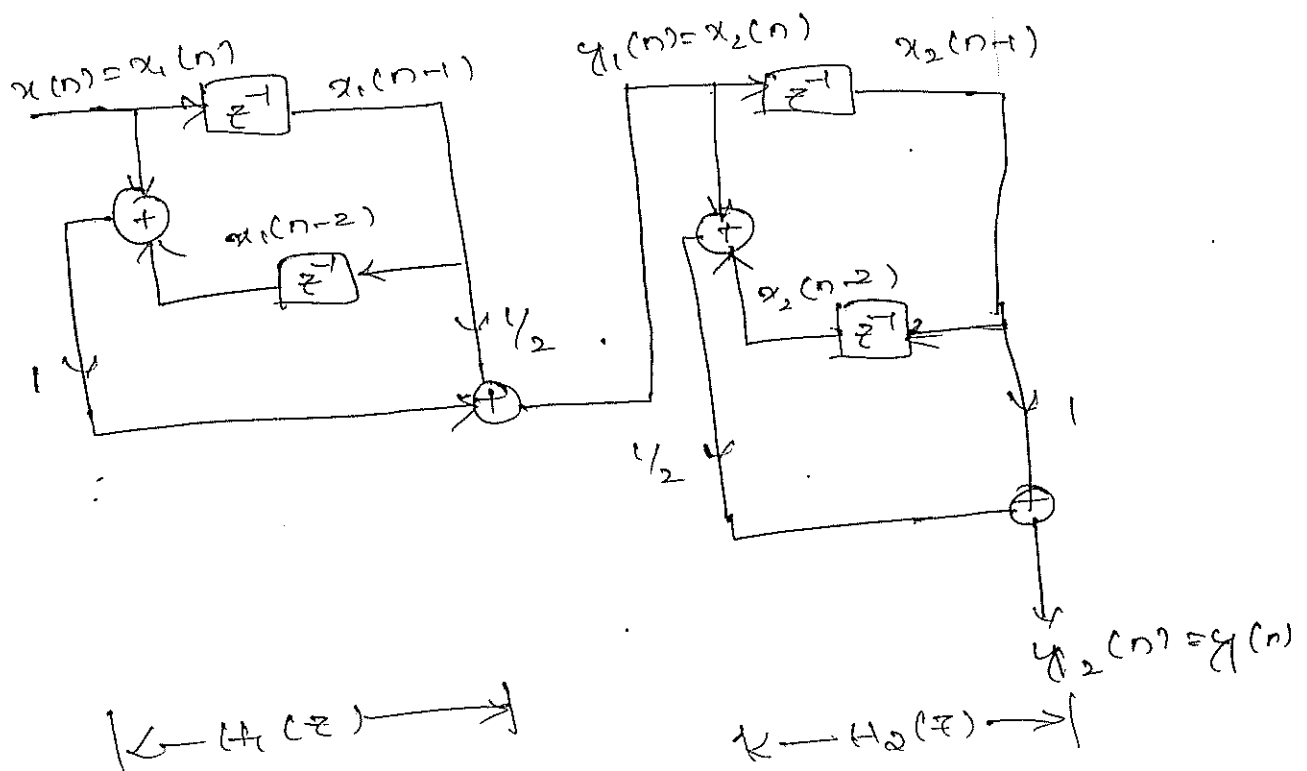
$$H_2(z) = \frac{Y_2(z)}{X_2(z)} = \frac{1}{2} + z^{-1} + \frac{1}{2} z^{-2}$$

$$Y_2(z) = \frac{1}{2} X_2(z) + z^{-1} X_2(z) + \frac{1}{2} z^{-2} X_2(z)$$

$$y_2(n) = \frac{1}{2} x_2(n) + x_2(n-1) + \frac{1}{2} x_2(n-2)$$

$$= \frac{1}{2} [x_2(n) + x_2(n-2)] + x_2(n-1)$$

②



4. FIR filters

A filter is freq. selective sys. Digital filters are classified as finite duration unit impulse response (FIR) filters ~~or~~ infinite duration unit impulse response (IIR) filters. depending on the form of unit impulse response of the sys. So in the FIR sys the impulse response sequence is of finite duration. The IIR sys has an infinite no. of non-zero terms. IIR filters are usually implemented using recursive structures (feedback poles & zeros) & FIR filters are usually implemented using non-recursive structures (no feedback & zeros).

The response of the FIR filter depends only on the present and past n samples. Whereas for the IIR filter the present response is a function of the present and past values of the excitation as well as past values of the response.

Advantages of FIR filters over IIR filters:-

- FIR filters are always stable.
- FIR filters with exactly linear phase can easily be designed.
- FIR filters can be realized in both recursive and non-recursive structures.
- FIR filters are free of limit cycle oscillation when implemented on a finite wordlength digital sys.

→ excellent design methods are available for various kinds of FIR filter.

Disadvantages of FIR filter are as follows:-

- The implementation of narrow transition band FIR filter is very costly as it requires more arithmetic operations & hardware components such as multipliers, adders, delay elements.
- Memory requirement & execution time are very high.

Comparison of IIR & FIR filter:-

IIR filter

FIR filter

→ All the infinite sample of impulse response are considered.

→ Only finite no. of samples of impulse response are considered.

→ The impulse response can't be directly converted to digital filter T.F.

→ The impulse response can be converted to digital filter T.F.

→ Linear phase characteristic can't be achieved.

→ Linear phase filters can be easily designed.

→ IIR filters are easily realized recursively.

→ FIR filters can be realized recursively & non-recursively.

→ The specifications include the desired characteristics for magnitude response only.

→ The specification include the desired characteristics for both magnitude & phase response.

→ The design involves design of analog filter & then transforming analog filter to digital filter.

→ The digital filter can be directly design to achieve the desired specifications.

The round of noise in IIR $\rightarrow$ Errors due to round of noise are less serious in FIR filters mainly because FB is not used.

Characteristics of FIR filters with linear phase

The T.F. of a FIR causal filter is given by

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

where $h(n)$ is the impulse response of the filter.

Now the freq response [F.T of $h(n)$] is given by

$$H(\omega) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n} = H(e^{j\omega})$$

Since $H(\omega)$ is complex it can be expressed as

$$H(\omega) = |H(\omega)| e^{j\theta(\omega)}$$

where $|H(\omega)|$ is the magnitude response and

$$H(\omega) = |H(\omega)| e^{j\theta(\omega)}$$

Here $\theta(\omega)$ is phase response

We define the phase delay T_p & group delay T_g of a filter as

$$T_p = \frac{-\theta(\omega)}{\omega}, \quad T_g = -\frac{d\theta(\omega)}{d\omega}$$

For FIR filters with linear phase we can define

$$\theta(\omega) = -\alpha\omega; \quad -\pi \leq \omega \leq \pi$$

$$T_p = \frac{+\alpha\omega}{\omega} = +\alpha, \quad T_g = -\frac{d(-\alpha\omega)}{d\omega} = +\alpha$$

Here $T_p = T_g = \alpha$ which means that α is independent of frequency.

$$H(\omega) = |H(\omega)| e^{j\theta(\omega)}$$

$$\sum_{n=0}^{N-1} h(n) e^{-j\omega n} = \pm |H(\omega)| e^{-j\alpha\omega}$$

$$\sum_{n=0}^{N-1} h(n) [\cos\omega n - j \sin\omega n] = \pm |H(\omega)| [\cos\alpha\omega - j \sin\alpha\omega] \quad \text{--- (1)}$$

On equating the real part & imaginary part of the above eq'n we get

$$\sum_{n=0}^{N-1} h(n) \cos\omega n = \pm |H(\omega)| \cos\alpha\omega \quad \text{--- (2)}$$

$$\sum_{n=0}^{N-1} h(n) \sin\omega n = \pm |H(\omega)| \sin\alpha\omega \quad \text{--- (3)}$$

On dividing eq'n (3) by (2)

$$\frac{\sum_{n=0}^{N-1} h(n) \sin\omega n}{\sum_{n=0}^{N-1} h(n) \cos\omega n} = \frac{\pm |H(\omega)| \sin\alpha\omega}{\pm |H(\omega)| \cos\alpha\omega}$$

$$\tan\omega n = \tan\alpha\omega$$

$$\sum_{n=0}^{N-1} h(n) \sin\omega n \cos\alpha\omega = \sum_{n=0}^{N-1} h(n) \sin\alpha\omega \cos\omega n$$

$$\sum_{n=0}^{N-1} h(n) \cos\omega n \sin\alpha\omega - \sum_{n=0}^{N-1} h(n) \sin\omega n \cos\alpha\omega = 0$$

$$\sum_{n=0}^{N-1} h(n) [\sin\alpha\omega \cos\omega n - \cos\alpha\omega \sin\omega n] = 0$$

$$\sum_{n=0}^{N-1} h(n) \sin(\alpha\omega - \omega n) = 0$$

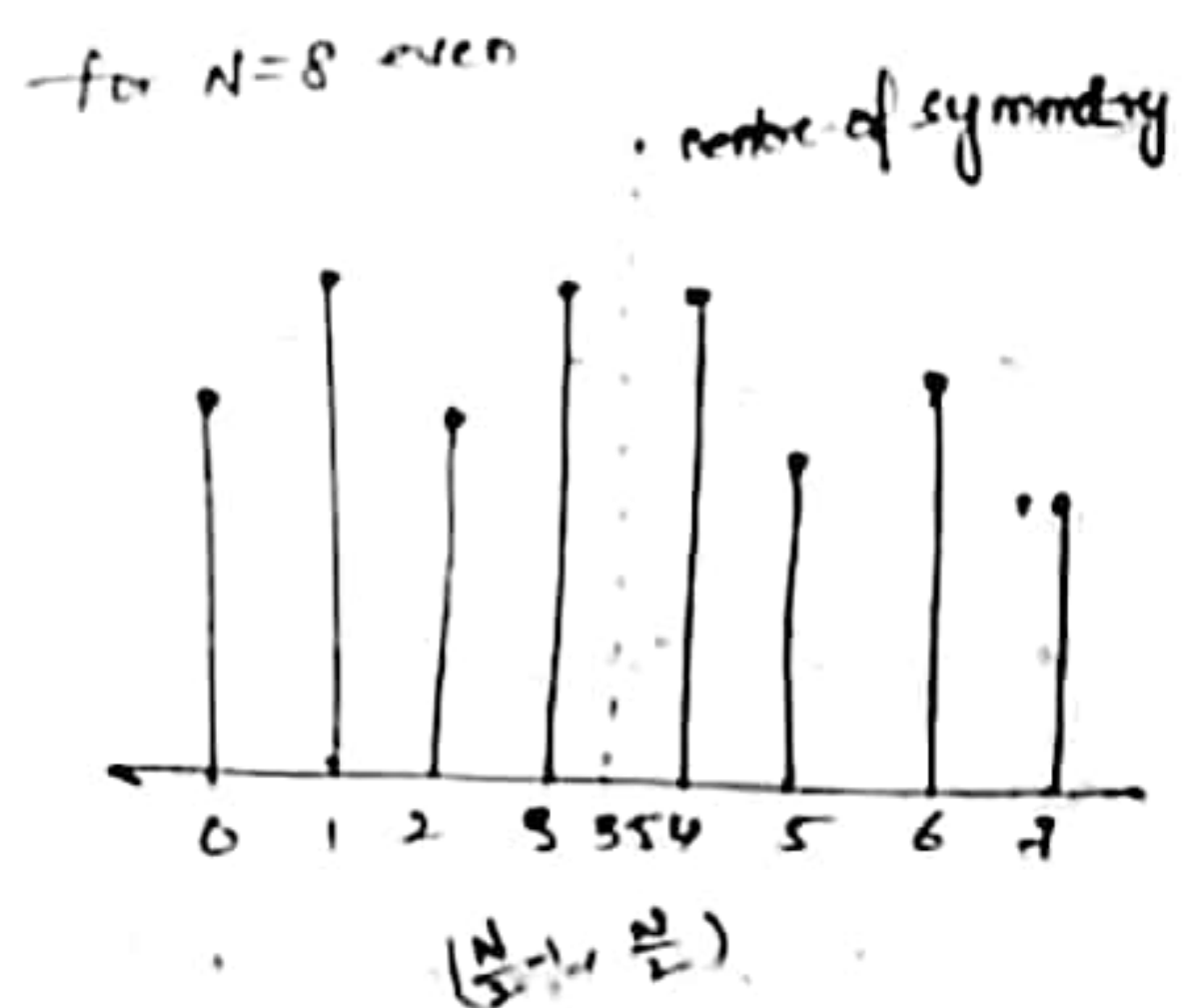
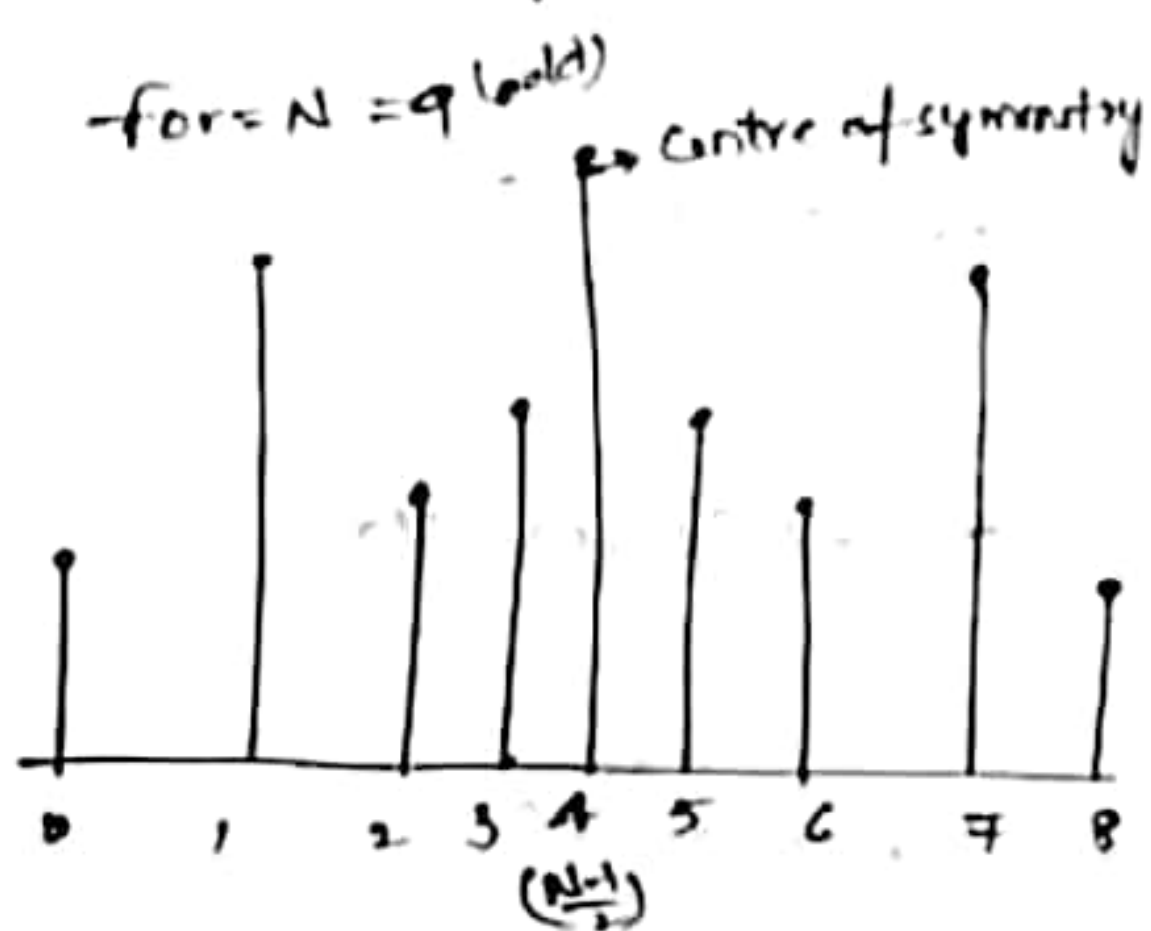
$$\boxed{\sum_{n=0}^{N-1} h(n) \sin(\alpha - n)\omega = 0} \quad \text{--- (4)}$$

One solution of eq'n (4) exists when $\alpha = \frac{N-1}{2}$

and $h(n) = h(N-1-n)$ for $0 \leq n \leq N-1$

$$\begin{aligned}
 & h(n) \sin(\alpha - n)\omega \\
 &= h(n) \sin\left(\left(\frac{N-1}{2} - n\right)\omega\right) \\
 &= h(n) \sin\left(\frac{N-1-n-n}{2} \omega\right) \\
 &= h(n) \sin\left(\frac{n-n}{2} \omega\right) \\
 &= 0.
 \end{aligned}$$

This shows that FIR filters will have constant phase and group delay when impulse response is symmetrical about $\alpha = \frac{N-1}{2}$. The impulse response satisfying the symmetric condition $h(n) = h(N-1-n)$ for odd, even values of N is shown in fig. when $N=9$ the center of the symmetry of sequence occurs at 4th sample & when $N=8$ the filter delay is $3\frac{1}{2}$ samples.



Case ii: If only constant group delay is required & not the phase delay

$$\theta(\omega) = \beta - \alpha\omega$$

$$\tau_p = -\frac{\theta(\omega)}{\omega} = -\frac{(\beta - \alpha\omega)}{\omega} = \alpha - \beta/\omega; \text{ It is not a constant}$$

$$\tau_g = -\frac{d\theta(\omega)}{d\omega} = -\frac{d(\beta - \alpha\omega)}{d\omega} = \alpha; \text{ It is constant}$$

$$H(\omega) = \pm |H(\omega)| e^{j\alpha(\omega)}$$

$$\Rightarrow \sum_{n=0}^{N-1} h(n) e^{-j\omega n} = \pm |H(\omega)| e^{j(\beta - \alpha\omega)}$$

$$\Rightarrow \sum_{n=0}^{N-1} h(n) [\cos \omega n - j \sin \omega n] = \pm |H(\omega)| [\cos(\beta - \alpha\omega) + j \sin(\beta - \alpha\omega)]$$

equating real and imaginary parts

$$\sum_{n=0}^{N-1} h(n) \cos \omega n = \pm |H(\omega)| \cos(\beta - \alpha\omega) \rightarrow (6)$$

$$-\sum_{n=0}^{N-1} h(n) \sin \omega n = \pm |H(\omega)| \sin(\beta - \alpha\omega) \rightarrow (7)$$

$$\text{Eq (7) } \Rightarrow \frac{-\sum_{n=0}^{N-1} h(n) \sin \omega n}{\sum_{n=0}^{N-1} h(n) \cos \omega n} = \frac{\pm |H(\omega)| \sin(\beta - \alpha\omega)}{\pm |H(\omega)| \cos(\beta - \alpha\omega)}$$

$$-\sum_{n=0}^{N-1} h(n) \sin \omega n [\cos(\beta - \alpha\omega)] = \sum_{n=0}^{N-1} h(n) \cos \omega n [\sin(\beta - \alpha\omega)]$$

$$\sum_{n=0}^{N-1} h(n) \cos \omega n \sin(\beta - \alpha\omega) + \sum_{n=0}^{N-1} h(n) \sin \omega n \cos(\beta - \alpha\omega) = 0$$

$$\sum_{n=0}^{N-1} h(n) [\cos \omega n \sin(\beta - \alpha\omega) + \sin \omega n \cos(\beta - \alpha\omega)] = 0 \quad \therefore \sin(A+B)$$

$$\sum_{n=0}^{N-1} h(n) \sin(\omega n + \beta - \alpha\omega) = 0$$

$$\boxed{\sum_{n=0}^{N-1} h(n) \sin(\omega(n-\alpha) + \beta) = 0} \quad \text{or} \quad \sum_{n=0}^{N-1} h(n) \sin(\beta - (\alpha-n)\omega) = 0$$

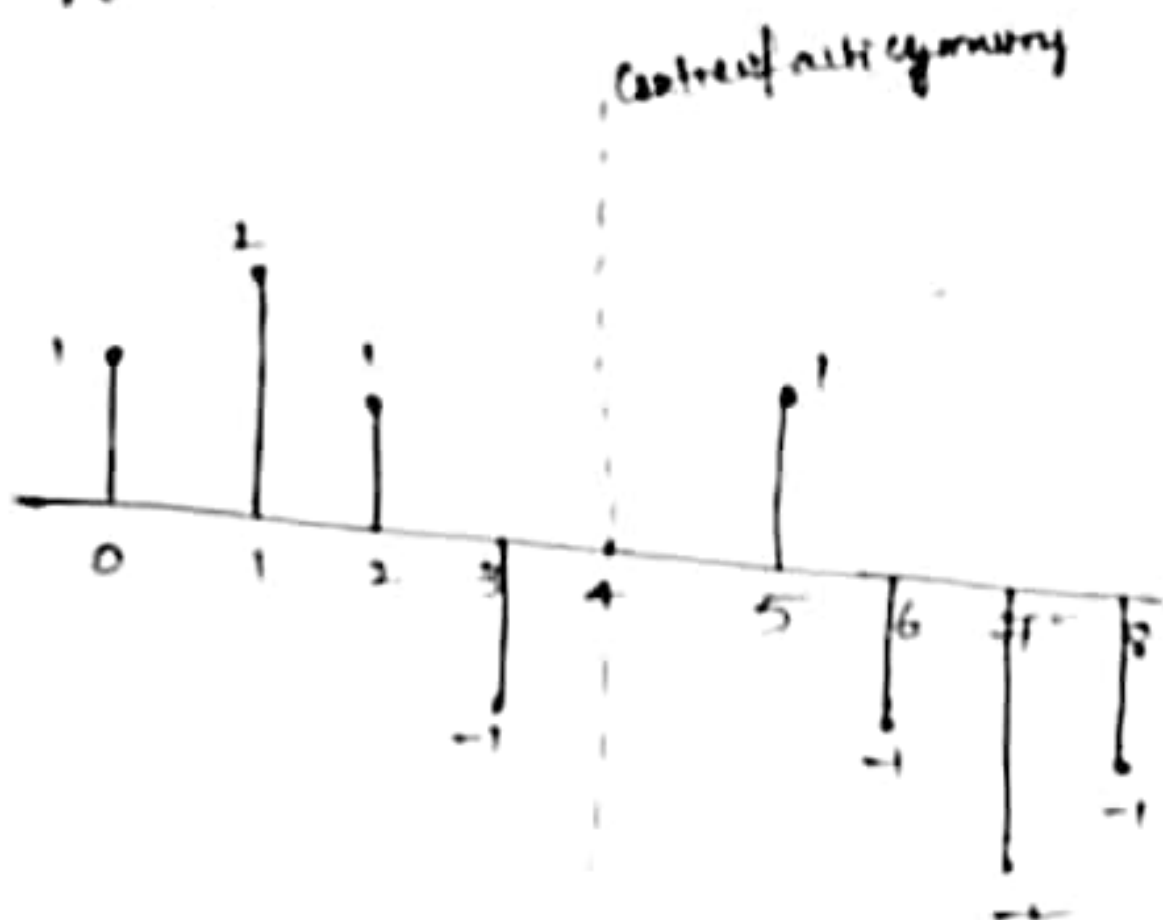
$$\text{If } \beta = \pi/2$$

$$\sum_{n=0}^{N-1} h(n) \sin(\pi/2 - (\alpha-n)\omega) = 0 \Rightarrow \boxed{\sum_{n=0}^{N-1} h(n) \cos(\alpha-n)\omega = 0}$$

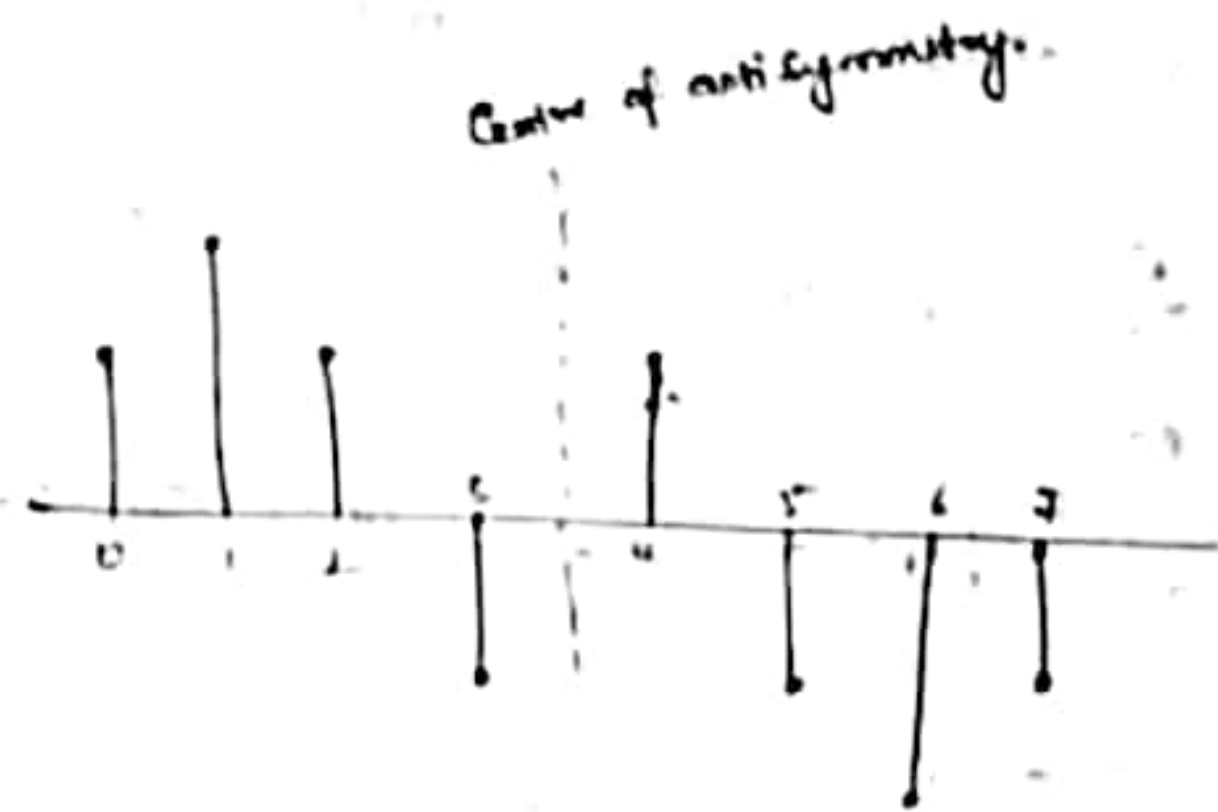
This eqn will be satisfy when $\alpha = \frac{N-1}{2}$ and $h(n) = h(N-1-n)$
 This shows that FIR filters have constant group delay $\tau_g = \alpha$ and
 τ_g is not constant when the impulse response is not symmetrical about
 $\alpha = \frac{N-1}{2}$.

Ex: The impulse response satisfying the anti symmetric condition is shown in the following figures when $N=9$ the centre of anti symmetry occurs at 4th sample and when $N=8$ the centre of anti symmetry occurs at $3\frac{1}{2}$ sample.

For $N=9$



For $N=8$



Frequency response of linear phase FIR filter:-

→ The frequency response of the filter is the F.T of its impulse response.

→ If $h(n)$ is the impulse response of the sy then the freq response of the sy is denoted by $H(e^{j\omega})(\omega)$ or $H(\omega)$.

→ $H(\omega)$ is a complex fun of freq ω and the can show it can be expressed as magnitude fun $|H(j\omega)|$ & phase fun $\angle H(j\omega)$.

→ Depending on the value of N (odd or even) and the type of symmetry response of the filter impulse response sequence (Symmetry / Anti symmetry), there are following 4 types of impulse response for linear phase FIR filter.

→ freq response of linear phase FIR filter when impulse response is symmetric and N is odd:-

Let $h(n)$ be the impulse response of the S/L. The
freq. response of the S/L $H(\omega)$ is given by

$$H(\omega) = H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n} = \sum_{n=0}^{N-1} h(n) e^{-j\omega n}$$

when N odd $h(n)$ is symmetric centre of symmetry
is at $\frac{N-1}{2}$

$$= \sum_{n=0}^{\frac{N-1}{2}} h(n) e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n) e^{-j\omega n}$$

$$m = N-1-n \quad ; \quad n = N-1-m$$

$$n = \frac{N+1}{2} \quad ; \quad m = N-1 - \frac{N+1}{2} = \frac{N-3}{2}$$

$$n = N-1 \quad ; \quad m = N-1 - (N-1) = 0$$

$$H(\omega) = \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(N-1-m) e^{-j\omega(N-1-m)}$$

put $m=n$

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(N-1-n) e^{-j\omega(N-1-n)}$$

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega(N-1-n)}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} e^{j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega(N-1-n)} e^{j\omega\left(\frac{N-1}{2}\right)} \right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n) \left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(N-1-n-\left(\frac{N-1}{2}\right)\right)} \right] \right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n) \left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{N-1}{2}-n\right)} \right] \right]$$

$$= e^{-j\omega(\frac{N-1}{2})} \left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n) \left[2 \cos \omega \left(\frac{N-1}{2} - n \right) \right] \right]$$

Let $k = \frac{N-1}{2} - n$; $n = \frac{N-1}{2} - k$.

$n=0$; $k = \frac{N-1}{2}$; $n = \frac{N-3}{2}$, $k = \frac{N-1}{2} - \frac{N-3}{2} = \frac{N-1-N+3}{2} = 1$.

$$= e^{-j\omega(\frac{N-1}{2})} \left[h\left(\frac{N-1}{2}\right) + \sum_{k=1}^{\frac{N-1}{2}} 2h\left[\frac{N-1}{2} - k\right] \cos \omega k \right]$$

put $k=n$.

$$H(\omega) = e^{-j\omega(\frac{N-1}{2})} \left[h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left[\frac{N-1}{2} - n\right] \cos \omega n \right]$$

The magnitude fun. of $H(\omega)$ is given by

$$|H(\omega)| = h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left[\frac{N-1}{2} - n\right] \cos \omega n$$

The phase fun. of $H(\omega)$ is given by

$$\angle H(\omega) = -\omega\left(\frac{N-1}{2}\right) = -\omega \tau$$

freq. response of Linear phase FIR filter when input response is symmetrical and N ^{EVEN} ~~odd~~:-

Proof:- freq. response of Linear phase FIR filter, with impulse response $h(n)$ & N samples.

$$H(\omega) = H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n}$$

for symmetric impulse response with even no. of samples the centre of symmetry lies b/n. $n = \frac{N-1}{2}$ & $n = \frac{N}{2}$

Hence $H(\omega)$ is expressed as

$$H(\omega) = \sum_{n=0}^{\frac{N}{2}-1} h(n) e^{-j\omega n} + \sum_{n=\frac{N}{2}}^{N-1} h(n) e^{-j\omega n}$$

Let $m = N-1-n$; $n = N-1-m$

$H(\omega) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n} + \sum_{m=0}^{N-1} h(N-1-m) e^{-j\omega(N-1-m)}$

$n = N-1$, $m = N-1 - N+1 = 0$

put $m=1$

$$H(\omega) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n} + \sum_{m=0}^{N-1} h(N-1-m) e^{-j\omega(N-1-m)}$$

$$= \sum_{n=0}^{N-1} h(n) e^{-j\omega n} + \sum_{n=0}^{N-1} h(N-1-n) e^{-j\omega(N-1-n)}$$

The impulse response is symmetric $h(N-1-n) = h(n)$

$$H(\omega) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n} + \sum_{n=0}^{N-1} h(n) e^{-j\omega(N-1-n)}$$

$$= e^{-j\omega(\frac{N-1}{2})} \left[\sum_{n=0}^{\frac{N-1}{2}-1} h(n) e^{-j\omega n} e^{j\omega(\frac{N-1}{2})} + \sum_{n=0}^{\frac{N-1}{2}-1} h(n) e^{-j\omega(N-1-n)} e^{j\omega(\frac{N-1}{2})} \right]$$

$$= e^{-j\omega(\frac{N-1}{2})} \sum_{n=0}^{\frac{N-1}{2}-1} h(n) \left[e^{+j\omega(\frac{N-1}{2}-n)} + e^{-j\omega(N-1-n-(\frac{N-1}{2}))} \right]$$

$$= e^{-j\omega(\frac{N-1}{2})} \sum_{n=0}^{\frac{N-1}{2}-1} h(n) \left[e^{j\omega(\frac{N-1}{2}-n)} + e^{-j\omega(\frac{N-1}{2}-n)} \right]$$

$$= e^{-j\omega(\frac{N-1}{2})} \sum_{n=0}^{\frac{N-1}{2}-1} h(n) 2 \cos \omega \left[\frac{N-1}{2} - n \right]$$

$$= e^{-j\omega(\frac{N-1}{2})} \sum_{n=0}^{\frac{N-1}{2}-1} h(n) 2 \cos \left[\frac{N-1}{2} - n - \frac{1}{2} \right]$$

Let $k = \frac{N-1}{2} - n$; $n = \frac{N-1}{2} - k$

$n = 0$; $k = \frac{N-1}{2}$

$n = \frac{N-1}{2} - 1$; $k = \frac{N-1}{2} - \frac{N-1}{2} + 1 = 1$

Put $k=n$.

$$H(\omega) = e^{-j\omega(\frac{N-1}{2})} \left[\sum_{n=1}^{\frac{N-1}{2}} h\left(\frac{N-1}{2}-n\right) 2 \cos \left[n - \frac{1}{2} \right] \right]$$

$$|H(\omega)| = \sum_{n=0}^{\frac{N-1}{2}} 2 h\left(\frac{N-1}{2}-n\right) \cos \omega \left[n - \frac{1}{2} \right]$$

$\angle H(\omega) = -\omega \left(\frac{N-1}{2} \right) = -\omega \alpha$ where $\alpha = \frac{N-1}{2}$

frequency response of Linear phase FIR filter when impulse response is antisymmetric and N is odd:-

Proof: freq. response of linear phase FIR filter with impulse response $h(n)$ of N -samples

$$H(\omega) = H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n}$$

antisym.
for a impulse response with N odd samples, the centre of symmetry is at $\frac{N-1}{2}$.

$$-H(\omega) = \sum_{n=0}^{\frac{N-1}{2}} h(n) e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n) e^{-j\omega n}$$

If impulse response is anti symmetric $h\left(\frac{N-1}{2}\right) = 0$.

$$H(\omega) = \sum_{n=0}^{\frac{N-1}{2}} h(n) e^{-j\omega n} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n) e^{-j\omega n}$$

Let $m = N-1-n$, $n = N-1-m$.

$$n=0 \Rightarrow m = N-1$$

$$n = \frac{N-1}{2} \Rightarrow m = N-1 - \frac{N-1}{2} = \frac{N-1}{2}$$

$$n = N-1 \Rightarrow m = 0$$

$$H(\omega) = \sum_{n=0}^{\frac{N-1}{2}} h(n) e^{-j\omega n} + \sum_{m=0}^{\frac{N-1}{2}} h(N-1-m) e^{-j\omega(N-1-m)}$$

$$H(\omega) = \sum_{n=0}^{\frac{N-1}{2}} h(n) e^{-j\omega n} + \sum_{n=0}^{\frac{N-1}{2}} h(N-1-n) e^{-j\omega(N-1-n)}$$

The impulse response is anti symmetric $h(N-1-n) = -h(n)$

$$H(\omega) = \sum_{n=0}^{\frac{N-1}{2}} h(n) e^{-j\omega n} - \sum_{n=0}^{\frac{N-1}{2}} h(n) e^{-j\omega(N-1-n)}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left[\sum_{n=0}^{\frac{N-1}{2}} h(n) \left[e^{+j\omega\left(\frac{N-1}{2} - n\right)} - e^{-j\omega\left(N-1-n - \left(\frac{N-1}{2}\right)} \right] \right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{\frac{N-1}{2}} h(n) \left[e^{+j\omega\left(\frac{N-1}{2} - n\right)} - e^{-j\omega\left(\frac{N-1}{2} - n\right)} \right]$$

Let $k = \frac{N-1}{2} - n$; $n = \frac{N-1}{2} - k$.

$$n=0 \Rightarrow k = \frac{N-1}{2}$$

$$n = \frac{N-1}{2} \Rightarrow k = \frac{N-1}{2} - \frac{N-1}{2} = 0$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{k=0}^{\frac{N-1}{2}} h\left(\frac{N-1}{2} - k\right) \left[2j \sin \omega \left(\frac{N-1}{2} - k \right) \right]$$

- system properties.
- Impulse response, step response.
- DIT & FFT
- Circular Convolution
- Difference b/w analog & digital filter,
- direct form 1.

$$|H(j\omega)| = e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - n\right) j \sin \omega n$$

$$j = e^{j\pi/2}$$

$$H(\omega) = e^{j\pi/2} e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - n\right) \sin \omega n$$

$$= e^{j\left[\frac{\pi}{2} - \omega\left(\frac{N-1}{2}\right)\right]} \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - n\right) \sin \omega n$$

$$|H(\omega)| = \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - n\right) \sin \omega n$$

$$\angle H(\omega) = \frac{\pi}{2} - \omega\left(\frac{N-1}{2}\right) = \beta - \omega\alpha$$

where $\beta = \frac{\pi}{2}$, $\alpha = \frac{N-1}{2}$

frequency response of Linear phase FIR filter when impulse response is -Antisymmetric and N is even:-

Proof:- freq. Response of Linear phase FIR filter with impulse response $h(n)$ is N sample.

$$H(\omega) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n}$$

Impulse response is -Anti symmetric and N - even
the centre of -Antisymmetry lies b/w $\frac{N}{2} - 1$ & $\frac{N}{2}$

$$H(\omega) = \sum_{n=0}^{\frac{N}{2}-1} h(n) e^{-j\omega n} + \sum_{n=\frac{N}{2}}^{N-1} h(n) e^{-j\omega n}$$

Let $m = N-1-n$; $n = N-1-m$

$$n = \frac{N}{2} ; m = N-1 - \frac{N}{2} = \frac{N}{2} - 1$$

$$H(\omega) = \sum_{n=0}^{\frac{N}{2}-1} h(n) e^{-j\omega n} + \sum_{m=0}^{\frac{N}{2}-1} h(N-1-m) e^{-j\omega(N-1-m)}$$

put $m = n$

$$H(\omega) = \sum_{n=0}^{\frac{N}{2}-1} h(n) e^{-j\omega n} + \sum_{n=0}^{\frac{N}{2}-1} h(N-1-n) e^{-j\omega(N-1-n)}$$

The impulse response is -Anti symmetric $h(N-1-n) = -h(n)$

$$H(\omega) = \sum_{n=0}^{\frac{N}{2}-1} h(n) e^{-j\omega n} + \sum_{n=0}^{\frac{N}{2}-1} -h(n) e^{-j\omega(N-1-n)}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{\frac{N}{2}-1} h(n) \left[e^{j\omega\left(\frac{N-1}{2}-n\right)} - e^{-j\omega\left(N-1-n-\left(\frac{N-1}{2}\right)\right)} \right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{\frac{N}{2}-1} h(n) \left[e^{j\omega\left(\frac{N-1}{2}-n\right)} - e^{-j\omega\left(\frac{N-1}{2}-n\right)} \right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{\frac{N}{2}-1} h(n) 2j \sin\omega\left[\frac{N-1}{2}-n\right]$$

$$H(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{\frac{N}{2}-1} h(n) 2j \sin\left(\frac{N}{2} - n - \frac{1}{2}\right)$$

Let $k = \frac{N}{2} - n \Rightarrow n = \frac{N}{2} - k$

$n = 0 ; k = \frac{N}{2}$

$n = \frac{N}{2} - 1 ; k = 1$

$$H(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{k=1}^{\frac{N}{2}} 2h\left(\frac{N}{2} - k\right) j \sin\left(k - \frac{1}{2}\right)$$

$$H(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=1}^{\frac{N}{2}} 2h\left(\frac{N}{2} - n\right) e^{j\frac{\pi}{2}} \sin\left(n - \frac{1}{2}\right)$$

$$H(\omega) = e^{j\left[\frac{\pi}{2} - \omega\left(\frac{N-1}{2}\right)\right]} \sum_{n=1}^{\frac{N}{2}} 2h\left(\frac{N}{2} - n\right) \sin\left(n - \frac{1}{2}\right)$$

$$\therefore |H(\omega)| = \sum_{n=1}^{\frac{N}{2}} 2h\left(\frac{N}{2} - n\right) \sin\left(n - \frac{1}{2}\right)$$

$$\angle H(\omega) = \frac{\pi}{2} - \omega\left(\frac{N-1}{2}\right) = \beta - \omega\alpha$$

where $\beta = \frac{\pi}{2}$, $\alpha = \frac{N-1}{2}$

Design Techniques for Linear phase FIR filter:

The well known methods of designing FIR filter are as follows.

1. Fourier series method.
2. Window method.
3. Frequency sampling method.

Fourier series method:-

The freq. response $H(e^{j\omega})$ or $H(\omega)$ of a system is periodic in ω . From F.S. analysis we know that any periodic fn can be expressed as a Linear Combination of complex exponentials.

$\therefore$ The desired freq. response of an FIR filter can be expressed by the f.s.

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n) e^{-j\omega n} \quad \text{--- (1)}$$

where the Fourier coefficients $h_d(n)$ are the desired impulse response sequence of the system. The samples of $h_d(n)$ can be determined by using the eq'n (which is inverse f.t. $H(e^{j\omega})$)

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega \quad \text{--- (2)}$$

The impulse response $h_d(n)$ from eq'n (2) is an infinite duration sequence.

For FIR filters we truncate this infinite impulse response to a finite duration sequence of length N , where N is odd

$$h(n) = \begin{cases} h_d(n) & \text{for } n = -\left(\frac{N-1}{2}\right) \text{ to } \left(\frac{N-1}{2}\right) \\ 0 & \text{otherwise} \end{cases}$$

Taking z.T. of $h(n)$ we get

$$z.T[h(n)] = H(z) = \sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} h(n) z^{-n}$$
$$= \sum_{n=-\frac{N-1}{2}}^{-1} h(n) z^{-n} + h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n) z^{-n} \quad \text{--- (3)}$$

$$= \sum_{n=1}^{\frac{N-1}{2}} h(n) z^n + h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n) z^{-n} \quad \text{--- (4)}$$

Impulse response is symmetric

$$h(-n) = h(n)$$
$$H(e^{j\omega}) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n) [e^{jn\omega} + e^{-jn\omega}] \quad \text{--- (5)}$$

This transfer fun of the filter $H(e^{j\omega})$ represents a non-causal filter (due to the presence of +ve power subject).

So, the transfer fun $H(e^{j\omega})$ is not physically realizable.

Realizability can be brought by multiplying eqn (5) by $e^{-j\frac{N-1}{2}\omega}$ where $\frac{N-1}{2}$ is delay in samples.

Hence we see that causality is brought about by multiplying the transfer fun by the delay factor $\alpha = e^{-j\frac{N-1}{2}\omega}$.

This modification doesn't affect the amplitude response of the amplitude response of the filter.

However the abrupt truncation of the FS results in oscillations in the pass & stop band.

These oscillations are due to the slow convergence of the partials near the point of discontinuity. This effect is known as Gibbs phenomenon. The undesirable oscillations can be reduced by multiplying the desired freq. response coefficient by an appropriate window fun.

Design procedure for ^{digital} FIR filter by Fourier series method:

→ The specification of Digital FIR filter are

* The desired freq. response $H_d(j\omega)$

* The Cut-off freq. ω_c for Lowpass & HPF and

ω_{c1} & ω_{c2} for bandpass & band stop filters.

Note:- If analog filter cut-off freq. F_c and sample

freq. F_s are specified then calculate the cut-off

Freq. of digital filter ω_c using the eqn $\boxed{\omega_c = \frac{2\pi f_c}{f_s}}$

* The no. of samples of impulse response 'N'

→ Determine the desired impulse response $h_d(n)$ by

$h_d(n) = \frac{1}{2\pi}$ taking inverse f.t of the desired

freq. response $H_d(e^{j\omega})$

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) d\omega$$

→ Calculate N samples of $h_d(n)$ for $n = -\left(\frac{N-1}{2}\right)$ to $\left(\frac{N-1}{2}\right)$

And from the impulse response $h(n) = h_d(n)$ with $n=0$ and so

Note: The impulse response is symmetric $h(-n) = h(n)$

Hence it is sufficient we calculate $h(n)$ for $n=0$ to $\frac{N-1}{2}$

→ Take z.t of impulse response to get the non-causal

transfer function of FIR filter $H(z) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} h(n) z^{-n}$

$$= h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n) (z^{-n} + z^n)$$

→ Convert the non-causal T.F $H(z)$ to causal

T.F $H'(z)$ by multiplying $H(z)$ by $z^{-\left(\frac{N-1}{2}\right)}$

$$H'(z) = z^{-\left(\frac{N-1}{2}\right)} \left[h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n) (z^{-n} + z^n) \right]$$

→ Draw a suitable structure for realization of

FIR filter.

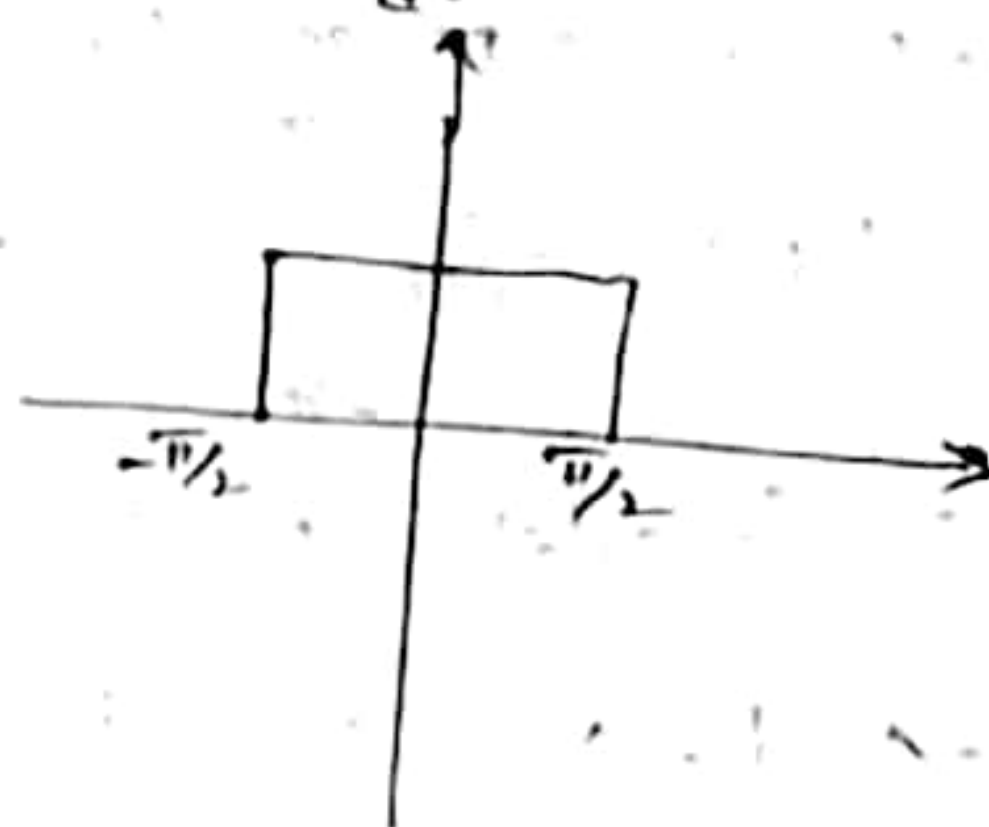
* Design an ideal LPF with freq. response

$$H_d(e^{j\omega}) = \begin{cases} 1 & \text{for } -\pi/2 \leq \omega < \pi/2 \\ 0 & \text{for } \pi/2 \leq |\omega| \leq \pi \end{cases} \quad \text{find the values}$$

of $h(n)$ for $N=11$, find $H(e^{j\omega})$ & plot the magnitude response.

Sol:-

$$H_d(e^{j\omega}) = \begin{cases} 1 & \text{for } -\pi/2 \leq \omega < \pi/2 \\ 0 & \text{for } \pi/2 \leq |\omega| \leq \pi \end{cases}$$



$$N=11$$

$$h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{j\omega n} d\omega$$

$$\alpha = \frac{N-1}{2}$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\omega n}}{jn} \right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\frac{\pi n}{2}} - e^{-j\frac{\pi n}{2}}}{jn} \right]$$

$$h(n) = \frac{1}{\pi n} \sin\left(\frac{\pi n}{2}\right)$$

$$n = -\alpha \text{ to } \alpha$$

$$h(n) = h(n) \quad \left| \quad n = -\left(\frac{N-1}{2}\right) \text{ to } \left(\frac{N-1}{2}\right) = -5 \text{ to } 5 \right.$$

$$h(n) = h(-n)$$

$$h(0) = \frac{0}{0} = \text{underdetermined}$$

By using L. Hospital rule

$$\left(\begin{array}{l} \lim_{n \rightarrow 0} \frac{\sin \pi n}{\pi n} = 1 \\ \lim_{n \rightarrow 0} \frac{\sin \pi n}{\pi n} = 1 \end{array} \right)$$

$$h(0) = h_d(0) = \lim_{n \rightarrow 0} \frac{\sin \frac{\pi n}{L}}{\pi n} = \frac{1}{2}$$

$$h(1) = \frac{\sin \frac{\pi}{2}}{\pi} = \frac{1}{\pi} = 0.318 = h(-1)$$

$$h(2) = \frac{\sin \pi}{2\pi} = 0 = h(-2)$$

$$h(3) = \frac{\sin \frac{3\pi}{2}}{3\pi} = \frac{-1}{3\pi} = -0.106 = h(-3)$$

$$h(4) = \frac{\sin 2\pi}{4\pi} = 0 = h(-4)$$

$$h(5) = \frac{\sin \left(\frac{5\pi}{2}\right)}{5\pi} = 0.0636 = h(-5)$$

$$H(z) = h(0) + \sum_{n=1}^{N-1} h(n) [z^n + z^{-n}]$$

$$= \frac{1}{2} + h(1)[z^1 + z^{-1}] + h(2)[z^2 + z^{-2}] + h(3)[z^3 + z^{-3}]$$

$$+ h(4)[z^4 + z^{-4}] + h(5)[z^5 + z^{-5}]$$

$$= \frac{1}{2} + 0.318(z^1 + z^{-1}) + 0.106(z^3 + z^{-3}) + 0.0636(z^5 + z^{-5})$$

$$H'(z) = z^{-5} H(z)$$

$$= z^{-5} H(z)$$

$$= z^{-5} \left[0.5 + 0.318(z^1 + z^{-1}) + 0.106(z^3 + z^{-3}) + 0.0636(z^5 + z^{-5}) \right]$$

$$= 0.5z^{-5} + 0.318z^{-4} + 0.318z^{-6} - 0.106z^{-2} - 0.106z^{-8}$$

$$+ 0.0636 + 0.0636z^{-10}$$

$$= 0.0636 - 0.106z^{-2} + 0.318z^{-4} + 0.5z^{-5} + 0.318z^{-6}$$

$$- 0.106z^{-8} + 0.0636z^{-10}$$

$h(n) = \text{filter coefficients}$

$$h(0) = h(10) = 0.0636$$

$$h(1) = h(9) = 0$$

$$h(2) = h(8) = -0.106$$

$$h(3) = h(7) = 0$$

$$h(4) = h(6) = 0.318$$

$$h(5) = h(5) = 0.5$$

$$h(n) = h(11-n)$$

magnitude response of $H(e^{j\omega})$

odd $N=11$

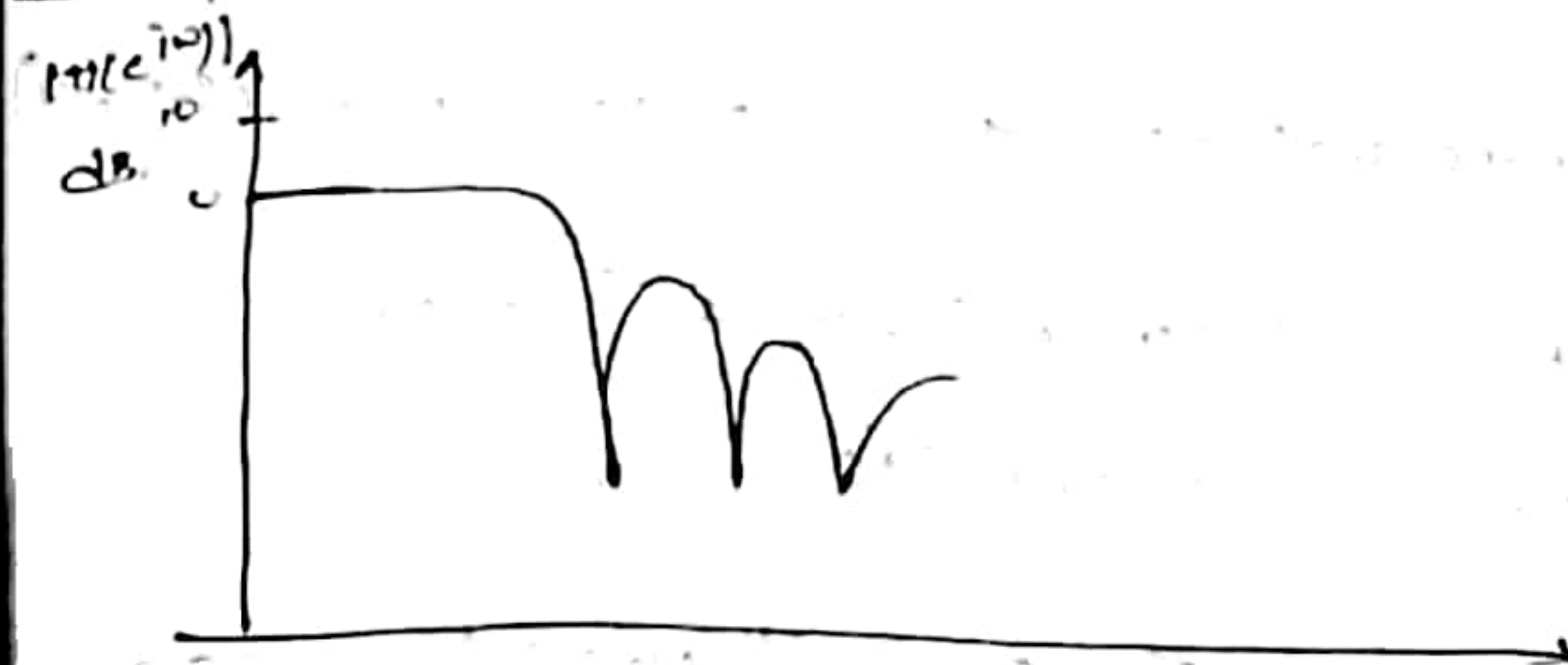
$$|H(e^{j\omega})| = |H(\omega)| = h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - n\right) \cos n\omega$$

$$|H(e^{j\omega})| = h(5) + 2h(4) \cos \omega + 2h(3) \cos 2\omega + 2h(2) \cos 3\omega + 2h(1) \cos 4\omega + 2h(0) \cos 5\omega$$

$$= 0.5 + 2(0.318) \cos \omega + 2(-0.106) \cos 2\omega + 2(0.0636) \cos 5\omega$$

$$= 0.5 + 0.636 \cos \omega - 0.212 \cos 2\omega + 0.1272 \cos 5\omega$$

ω	0	10	20	30	40	50	60	70	80	90	100	110	120
$ H(e^{j\omega}) $	0.054	1.07	-0.21	-0.212	0.77	0.21	1.77	-6	-14.56	31.8	-20.6		



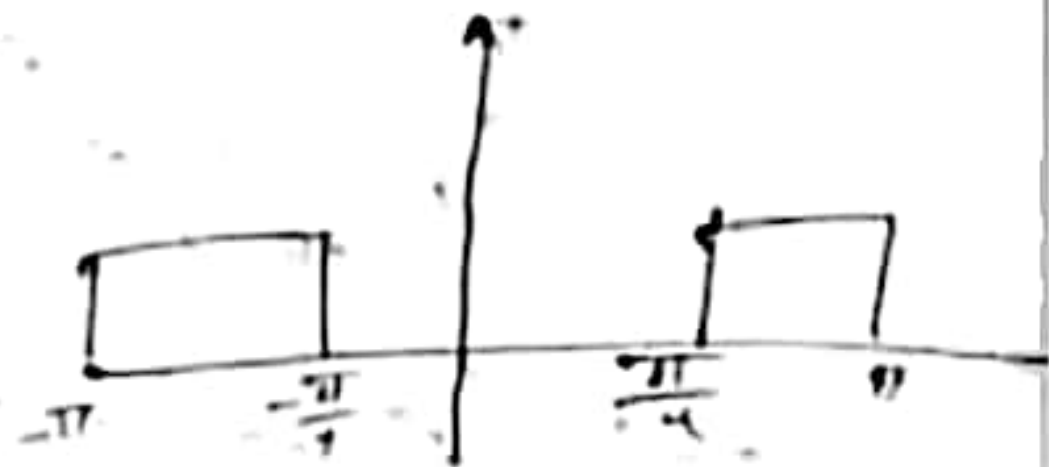
* Design an ideal TPF with freq. response $H_d(e^{j\omega}) = 1$

1 for $\frac{\pi}{4} \leq |\omega| \leq \pi$. find the values of $h(n)$ for $n=0$

0 for $|\omega| \leq \frac{\pi}{4}$. find $\tau(\omega)$. plot the magnitude response.

sol:-

$$h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega$$



$$= \frac{1}{2\pi} \int_{-\pi}^{-\pi/4} e^{j\omega n} d\omega + \frac{1}{2\pi} \int_{\pi/4}^{\pi} e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\omega n}}{jn} \right]_{-\pi}^{-\pi/4} + \frac{1}{2\pi} \left[\frac{e^{j\omega n}}{jn} \right]_{\pi/4}^{\pi}$$

$$= \frac{1}{2\pi} \left[\frac{e^{-j\pi n/4}}{jn} - \frac{e^{-j\pi n}}{jn} \right] + \frac{1}{2\pi} \left[\frac{e^{j\pi n}}{jn} - \frac{e^{j\pi n/4}}{jn} \right]$$

$$= \frac{1}{2\pi jn} \left[e^{-j\pi n/4} - e^{-j\pi n} \right] + \frac{1}{2\pi jn} \left[e^{j\pi n} - e^{j\pi n/4} \right]$$

$$= \frac{1}{2\pi jn} \left[e^{j\pi n} + e^{-j\pi n/4} - e^{j\pi n/4} - e^{-j\pi n} \right]$$

$$= \frac{1}{2\pi jn} \left[2j \sin \pi n + 2j \sin \left(\frac{\pi n}{4} \right) \right]$$

$$= \frac{1}{\pi n} \left[\sin \pi n + \sin \left(\frac{\pi n}{4} \right) \right] \quad n = -\infty \text{ to } \infty$$

$$h(n) = h(n) \Big|_{n = -\left(\frac{N-1}{2}\right) \text{ to } \left(\frac{N-1}{2}\right) = -r \text{ to } r}$$

$$h(n) = h(-n)$$

$$\therefore h(0) = \frac{1}{\pi(0)} \sin = \lim_{n \rightarrow 0} \left[\frac{\sin \pi n + \sin \left(\frac{\pi n}{4} \right)}{n\pi} \right]$$

$$h(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega\theta} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{-\pi/4} 1 d\omega + \frac{1}{2\pi} \int_{-\pi/4}^{\pi} d\omega$$

$$= \frac{1}{2\pi} \left[\omega \Big|_{-\pi}^{-\pi/4} \right] + \frac{1}{2\pi} \left[\omega \Big|_{-\pi/4}^{\pi} \right]$$

$$= \frac{1}{2\pi} \left[-\frac{\pi}{4} + \pi \right] + \frac{1}{2\pi} \left[\pi - \left(-\frac{\pi}{4}\right) \right] = \frac{3}{4}$$

$$h(1) = h(-1) = \frac{1}{\pi} \left[\sin \pi - \sin \frac{\pi}{4} \right] = -0.225$$

$$h(2) = h(-2) = -0.159$$

$$h(3) = h(-3) = -0.075$$

$$h(4) = h(-4) = -0.045 = 0$$

$$h(5) = h(-5) = 0.045$$

$$H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n) (z^n + z^{-n})$$

$$= \frac{3}{4} + h(1) [z + z^{-1}] + h(2) [z^2 + z^{-2}] + h(3) [z^3 + z^{-3}]$$

$$+ h(4) [z^4 + z^{-4}] + h(5) [z^5 + z^{-5}]$$

$$= \frac{3}{4} - 0.225 [z + z^{-1}] - 0.159 [z^2 + z^{-2}] - 0.075 [z^3 + z^{-3}]$$

$$+ 0.045 [z^5 + z^{-5}]$$

Realizable filter T.F.:-

$$H'(z) = z^{-\left(\frac{N-1}{2}\right)} H(z)$$

$$= z^{-5} [H(z)]$$

$$= 0.75z^{-5} - 0.225z^{-4} - 0.225z^{-6} - 0.159z^{-3} - 0.159z^{-7} - 0.075z^{-8}$$

$$- 0.075z^{-8} + 0.045 + 0.045z^{-10}$$

$$= 0.045 - 0.075z^{-2} - 0.159z^{-3} - 0.225z^{-4} + 0.75z^{-5}$$

$$- 0.225z^{-6} - 0.159z^{-7} - 0.075z^{-8} + 0.045z^{-10}$$

Causal filter coefficients are

$$h(0) = h(10) = 0.095$$

$$h(1) = h(9) = 0$$

$$h(2) = h(8) = -0.095$$

$$h(3) = h(7) = -0.159$$

$$h(4) = h(6) = -0.225$$

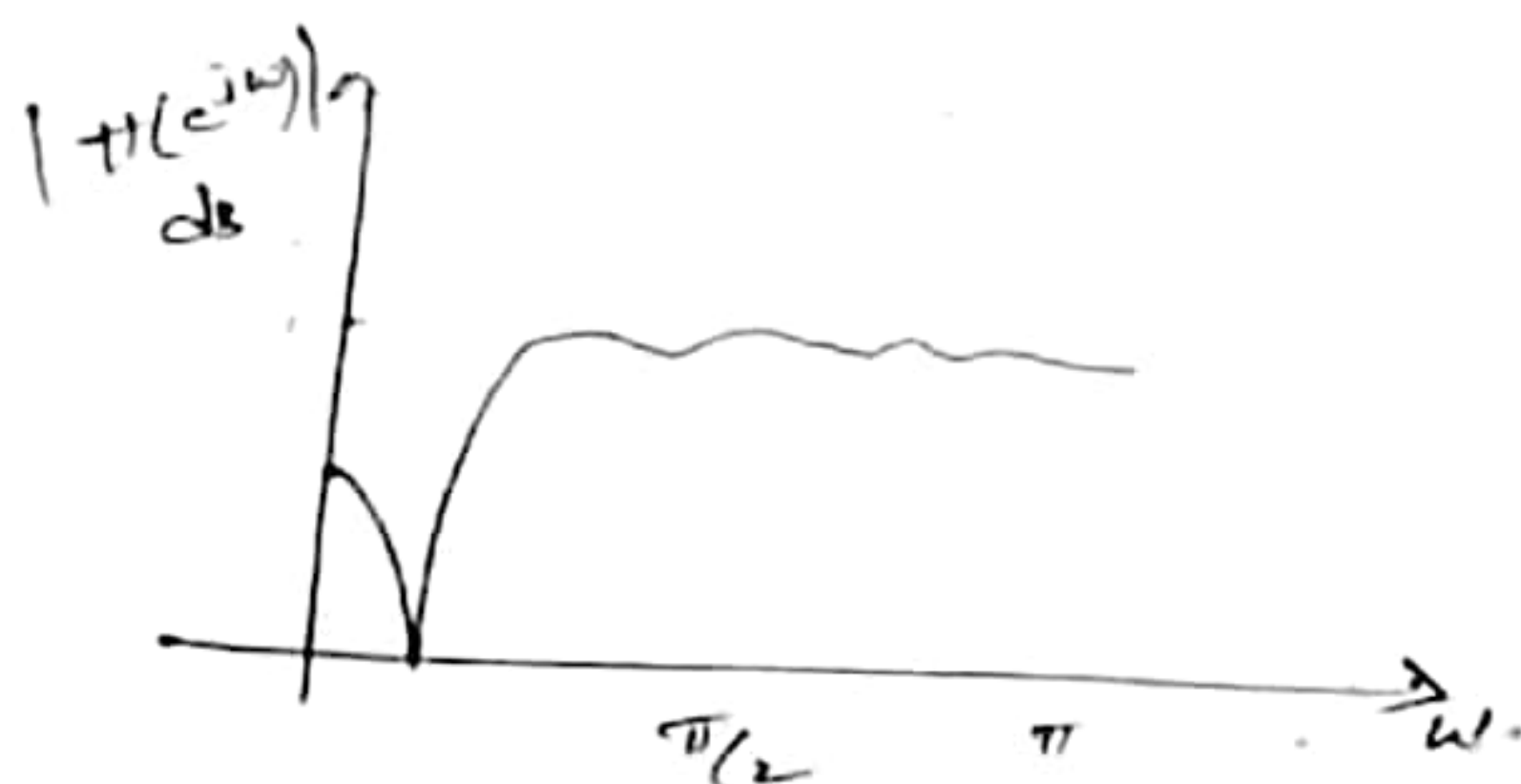
$$h(5) = 0.75$$

$$|H(e^{j\omega})| = h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - n\right) \cos n\omega$$

$$= h(5) + 2h(4) \cos \omega + 2h(3) \cos 2\omega + 2h(2) \cos 3\omega + 2h(1) \cos 4\omega + 2h(0) \cos 5\omega$$

$$= 0.75 + 0.45 \cos \omega - 0.318 \cos 2\omega - 0.15 \cos 3\omega + 0.09 \cos 4\omega$$

ω	0	10	20	30	40	50	60	70	80	90	100
$ H(e^{j\omega}) $	-22dB	-23.62	-46.3	-18.2	-9.16	-4.2	-1.1	-0.504	0.95	0.57	-0.15



* Design an ideal band pass filter with freq.

response $H_d(e^{j\omega}) = \begin{cases} 1 & \text{for } \frac{\pi}{4} \leq |\omega| \leq \frac{3\pi}{4} \\ 0 & \text{otherwise} \end{cases}$ find the

values of $h(n)$ for $n=11$ & plot freq. response.

sol:
$$h_d(n) = \frac{1}{2\pi} \int_{-\frac{3\pi}{4}}^{-\frac{\pi}{4}} e^{j\omega n} d\omega + \frac{1}{2\pi} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\omega n}}{jn} \right]_{-\pi/4}^{\pi/4} + \frac{1}{2\pi} \left[\frac{e^{j\omega n}}{jn} \right]_{\pi/4}^{3\pi/4}$$

$$h(n) = \frac{1}{2\pi} \left[\frac{e^{j\frac{n\pi}{4}} - e^{-j\frac{n\pi}{4}}}{jn} \right] + \frac{1}{2\pi jn} \left[e^{j\frac{3n\pi}{4}} - e^{j\frac{n\pi}{4}} \right]$$

$$= \frac{1}{2\pi jn} \left[e^{-j\frac{n\pi}{4}} - e^{-j\frac{3n\pi}{4}} + e^{j\frac{3n\pi}{4}} - e^{j\frac{n\pi}{4}} \right]$$

$$= \frac{1}{2\pi jn} \left[2j \sin\left(\frac{n\pi}{4}\right) - 2j \sin\left(\frac{n\pi}{4}\right) \right]$$

$$= \frac{1}{\pi n} \left[-\sin\left(\frac{3n\pi}{4}\right) + \sin\left(\frac{n\pi}{4}\right) \right]$$

$$h(n) = h(n) \Big|_{n = -(\frac{N-1}{2}) + 1}^{(\frac{N-1}{2})} = -5 + 0 + 5$$

$$h(0) = \lim_{n \rightarrow 0} \frac{1}{\pi n} \frac{\sin \frac{3n\pi}{4}}{\frac{3n\pi}{4} \cdot \frac{4}{3}} - \lim_{n \rightarrow 0} \frac{\sin \frac{n\pi}{4}}{\frac{n\pi}{4} \times 4}$$

$$= \frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

$$h(1) = h(-1) = \frac{1}{\pi} \left[\sin \frac{3\pi}{4} - \sin \frac{\pi}{4} \right] = 0$$

$$h(2) = h(-2) = -0.3183$$

$$h(3) = h(-3) = +0$$

$$h(4) = h(-4) = 0$$

$$h(5) = h(-5) = 0$$

$$H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n) (z^n + z^{-n})$$

$$= 0.5 + h(2) (z^2 + z^{-2}) = 0.5 - 0.3183 (z^{-2} + z^2)$$

$$H'(z) = z^{-\left(\frac{N-1}{2}\right)} H(z)$$

$$= z^{-5} (0.5 - 0.3183 (z^{-2} + z^2))$$

$$= 0.5 z^{-5} - 0.3183 z^{-7} - 0.3183 z^{-3}$$

$$= -0.31832^{-3} + 0.6366^{-3} 0.52^{-7} - 0.31832^{-7}$$

$$h(0) = h(10) = 0.5$$

$$h(1) = h(9) = 0$$

$$h(2) = h(8) = -0.31830$$

$$h(3) = h(7) = 0.3183$$

$$h(4) = h(6) = 0$$

$$h(5) = 0.5$$

$$|H(e^{j\omega})| = h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - n\right) \cos n\omega$$

$$= 0.5 + 2h(3) \cos 2\omega$$

$$|H(e^{j\omega})| = 0.5 - 0.6366 \cos 2\omega$$

ω	0	20	40	60	80	90	115	120	135	150	160	180
$20 \log H(e^{j\omega}) $	-17.3	-35.17	-14.8	-16.2	-1.94	0.4316	1.11	0.4316	-1.94	-6.02	-14.8	-35.17



* Design an ideal band reject filter with a desired

freq. response $H(e^{j\omega}) = 1$ for $|\omega| \leq \pi/3$ & $|\omega| \geq 2\pi/3$
 $= 0$ for otherwise.

find $h(n)$ for $N=11$. - find $H(z)$ plot the magnitude response.

$$\underline{\text{Sol:}} - h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega$$

$$|\omega| \leq \pi/3$$

$$\omega \leq \pi/3$$

$$-\omega \leq \pi/3$$

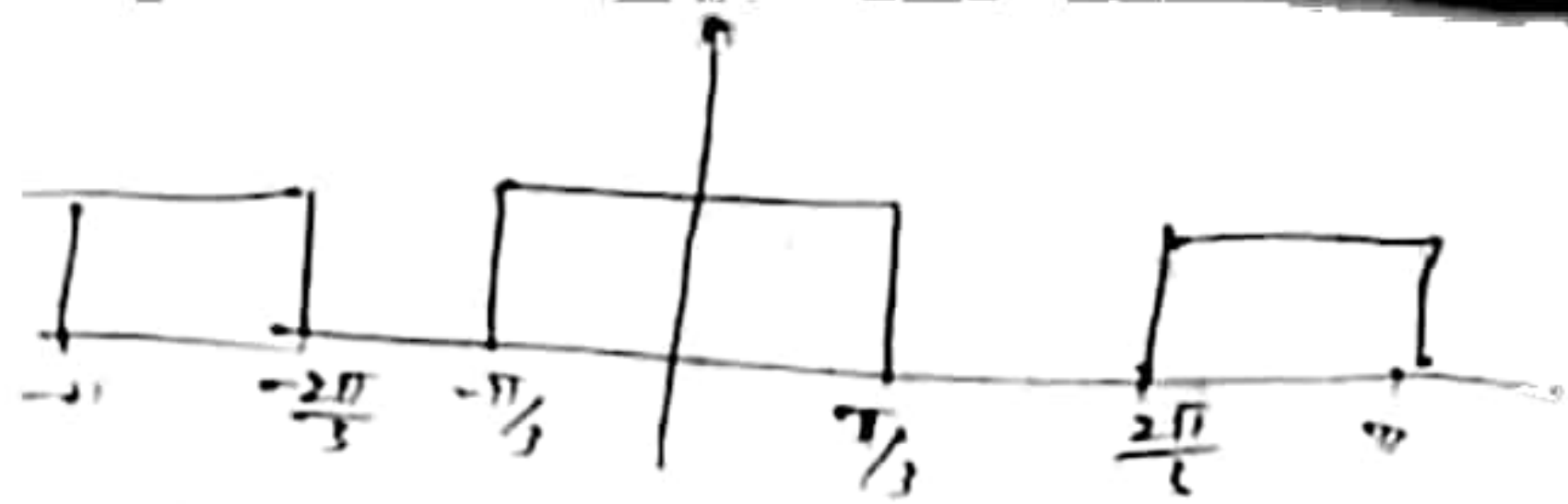
$$\omega \geq -\pi/3$$

$$|\omega| \geq 2\pi/3$$

$$\omega \geq 2\pi/3$$

$$-\omega \geq 2\pi/3$$

$$\omega \leq -2\pi/3$$



$$= \frac{1}{2\pi} \int_{-\pi}^{-2\pi/3} e^{j\omega n} d\omega + \frac{1}{2\pi} \int_{-\pi/3}^{\pi/3} e^{j\omega n} d\omega + \frac{1}{2\pi} \int_{2\pi/3}^{\pi} e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\omega n}}{jn} \right]_{-\pi}^{-2\pi/3} + \frac{e^{j\omega n}}{jn} \left[\frac{\pi/3}{-\pi/3} \right] + \frac{e^{j\omega n}}{jn} \left[\frac{\pi}{2\pi/3} \right]$$

$$= \frac{1}{2\pi jn} \left[\left(e^{j\pi n} - e^{-j\pi n} \right) + \left(e^{jn\pi/3} - e^{-jn\pi/3} \right) + \left(e^{jn\pi/3} - e^{-jn\pi/3} \right) \right]$$

$$= \frac{1}{\pi n} \left[\sin n\pi + \sin \frac{n\pi}{3} - \sin \frac{2n\pi}{3} \right]$$

Step 2 $h(n) = h(-n)$

$$N = 11$$

$$h(n) = h_d(n) \left[1 - \left(\frac{N-1}{2} \right) + \left(\frac{N+1}{2} \right) \right] = -5 \text{ for } n=0$$

$$h(0) = \lim_{n \rightarrow 0} \left[\frac{1}{\pi n} \left(\sin n\pi + \sin \frac{n\pi}{3} - \sin \frac{2n\pi}{3} \right) \right]$$

$$= 1 + \frac{1}{3} - \frac{2}{3} = \frac{2}{3}$$

$$h(1) = h(-1) = 0$$

$$h(2) = h(-2) = 0.285$$

$$h(3) = h(-3) = 0$$

$$h(4) = h(-4) = -0.319$$

$$h(5) = 0$$

Steps:- non-Casual.

$$H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n) [z^n + z^{-n}]$$

$$H(z) = h(0) + h(1) [z^{-1} + z^1] + h(2) [z^2 + z^{-2}]$$

$$+ h(3) [z^3 + z^{-3}] + h(4) [z^4 + z^{-4}] + h(5) [z^5 + z^{-5}]$$

$$H(z) = 0.66 + 0.275 [z^{-2} + z^2] + 0 - 0.317 [z^{-4} + z^4] + 0$$

$$H(z) = 0.66 + 0.275 [z^{-2} + z^2] - 0.317 [z^{-4} + z^4]$$

Step 4 Casual.

$$H'(z) = z^{-\left(\frac{N-1}{2}\right)} H(z)$$

$$H'(z) = z^{-5} [0.66 + 0.275 (z^{-2} + z^2) - 0.317 (z^{-4} + z^4)]$$

$$= -0.317 z^{-1} + 0.275 z^{-3} + 0.66 z^{-5} + 0.275 z^{-7} - 0.317 z^{-9}$$

Steps:- Casual filter Coefficient.

$$h(n) = h(N-1-n)$$

$$h(0) = h(10) = 0$$

$$h(1) = h(9) = -0.317$$

$$h(2) = h(8) = 0$$

$$h(3) = h(7) = 0.275$$

$$h(4) = h(6) = 0$$

$$h(5) = 0.66.$$

steps:- Magnitude Response

$$|H(e^{j\omega})| = h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - n\right) \cos n\omega$$

$$= h(5) + 2h(4)\cos\omega + 2h(3)\cos 2\omega + 2h(2)\cos 3\omega + 2h(1)\cos 4\omega + 2h(0)\cos 5\omega$$

$$|H(e^{j\omega})| = 0.66 + 0.55\cos\omega - 0.27\cos 4\omega$$

b. Design of FIR filters using window.

The procedure for designing FIR filter using window is

- Choose the desired freq. Response of the filter $H_d(e^{j\omega})$
- Take I.F.T of $H_d(e^{j\omega})$ to obtain the desired impulse Response $h_d(n)$
- Choose a window sequence $w(n)$ & multiply $h_d(n)$ by $w(n)$ to convert the infinite duration impulse response to finite duration

$$h(n) = h_d(n)w(n)$$

- The TF $H(e^{j\omega})$ of the filter is obtained by taking $\text{F.T}[h(n)]$.

Window Sequence for FIR filter design:-

Name of the window.

Window Sequence.

→ Rectangular window

$$w_R(n) = \begin{cases} 1; & -\frac{N-1}{2} \leq n \leq \frac{N-1}{2} \\ 0; & \text{otherwise} \end{cases}$$

→ Triangular window

$$w_T(n) = \begin{cases} \frac{1-2|n|}{N-1}; & -\frac{N-1}{2} \text{ to } \frac{N-1}{2} \\ 0; & \text{otherwise} \end{cases}$$

→ Hamming window

$$w_H(n) = \begin{cases} 0.54 + 0.46 \cos\left(\frac{2\pi n}{N-1}\right) & -\left(\frac{N-1}{2}\right) \leq n \leq \frac{N-1}{2} \\ 0 & ; \text{otherwise} \end{cases}$$

→ Hanning window

$$w_{Hn}(n) = \begin{cases} 0.5 + 0.5 \cos\left(\frac{2\pi n}{N-1}\right) & -\left(\frac{N-1}{2}\right) \leq n \leq \frac{N-1}{2} \\ 0 & ; \text{otherwise} \end{cases}$$

→ Blackman window

$$w_B(n) = \begin{cases} 0.42 + 0.5 \cos\left(\frac{2\pi n}{N-1}\right) + 0.08 \cos\left(\frac{4\pi n}{N-1}\right) & -\left(\frac{N-1}{2}\right) \leq n \leq \frac{N-1}{2} \\ 0 & ; \text{otherwise} \end{cases}$$

→ Kaiser window

$$w_K(n) = \frac{I_0\left(\alpha \sqrt{1 - \left(\frac{2n}{N-1}\right)^2}\right)}{I_0(\alpha)}$$

P. Design an ideal highpass filter with frequency response

$$H_d(e^{j\omega}) = \begin{cases} 1 & ; \frac{\pi}{4} \leq |\omega| \leq \pi \\ 0 & ; |\omega| \leq \frac{\pi}{4} \end{cases} \quad \text{find the value of } h(n) \text{ for } n=1$$

find $h(n)$ and plot the mag response having hanning window.

Sol: step 1..

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega$$

$$h_d(n) = \frac{1}{2\pi} \left[\int_{-\pi}^{-\pi/4} e^{j\omega n} d\omega + \int_{\pi/4}^{\pi} e^{j\omega n} d\omega + \int_{-\pi/4}^{\pi/4} 0 d\omega \right]$$

$$= \frac{1}{2\pi} \left[\int_{-\pi}^{-\pi/4} e^{j\omega n} d\omega + \int_{\pi/4}^{\pi} e^{j\omega n} d\omega \right]$$

$$h_d(n) = \frac{1}{2\pi} \left[\sin \omega n - \sin \frac{\pi n}{4} \right]$$

step 2 Hanning window

$$w_{hn}(n) = \begin{cases} 0.5 + 0.5 \cos \frac{2\pi n}{N-1} & ; -\frac{N-1}{2} \leq n \leq \frac{N-1}{2} \\ 0 & ; \text{otherwise} \end{cases}$$

$$w_{hn}(n) = \begin{cases} 0.5 + 0.5 \cos \frac{\pi n}{5} & ; -5 \leq n \leq 5 \\ 0 & ; \text{otherwise} \end{cases} \quad \boxed{N=11}$$

step 3 $h(n) = h_d(n) \cdot w_{hn}(n) \quad -5 \leq n \leq 5$

$$h_d(0) = \frac{3}{4} = 0.75$$

$$h_d(1) = h_d(-1) = -0.225$$

$$h_d(2) = h_d(-2) = -0.159$$

$$h_d(3) = h_d(-3) = -0.075$$

$$h_d(4) = h_d(-4) = 0$$

$$h_d(5) = h_d(-5) = 0.041$$

$$w_{hn}(0) = 1$$

$$\omega_{hn}(1) = \omega_{hn}(-1) = 0.5 + 0.5 \cos \pi/5 = 0.9045$$

$$\omega_{hn}(2) = \omega_{hn}(-2) = 0.5 + 0.5 \cos 2\pi/5 = 0.655$$

$$\omega_{hn}(3) = \omega_{hn}(-3) = 0.5 + 0.5 \cos 3\pi/5 = 0.345$$

$$\omega_{hn}(4) = \omega_{hn}(-4) = 0.5 + 0.5 \cos 4\pi/5 = 0.0945$$

$$\omega_{hn}(5) = \omega_{hn}(-5) = 0.5 + 0.5 \cos 5\pi/5 = 0$$

$$h(n) = h_d(n) \cdot \omega_{hn}(n)$$

$$h(0) = h_d(0) \cdot \omega_{hn}(0) = 0.75 \times 1 = 0.75$$

$$h(1) = h(-1) = h_d(1) \omega_{hn}(1) = -0.225 \times 0.9045 = -0.204$$

$$h(2) = h(-2) = h_d(2) \omega_{hn}(2) = -0.159 \times 0.655 = -0.104$$

$$h(3) = h(-3) = h_d(3) \omega_{hn}(3) = -0.071 \times 0.345 = -0.026$$

$$h(4) = h(-4) = h_d(4) \omega_{hn}(4) = 0 \times 0.0945 = 0$$

$$h(5) = h(-5) = h_d(5) \omega_{hn}(5) = 0.045 \times 0 = 0$$

Step 4: T.F of the following filter given by

$$H(z) = h(0) + \sum_{n=1}^5 h(n) [z^{-n} + z^n]$$

$$H(z) = 0.75 + h(1)[z^{-1} + z^1] + h(2)[z^{-2} + z^2] + h(3)[z^{-3} + z^3] \\ + h(4)[z^{-4} + z^4] + h(5)[z^{-5} + z^5]$$

$$= 0.75 - 0.204[z^{-1} + z^1] - 0.104[z^{-2} + z^2] - 0.026[z^{-3} + z^3]$$

$$= 0.75 - 0.204z^{-1} - 0.204z^{-1} - 0.104z^2 - 0.104z^2 - 0.104z^2 \\ - 0.026z^3 - 0.026z^3$$

Step 5: T.F of Realizable filter is

$$H'(z) = z^{-\left(\frac{N-1}{2}\right)} H(z) = z^{-5} H(z)$$

$$H'(z) = 0.75z^{-5} - 0.204z^{-6} - 0.204z^{-4} - 0.104z^{-7} - 0.104z^2 \\ - 0.026z^{-8} - 0.026z^{-2}$$

Step 6: Find filter coefficients

$$h(n) = h(N-1-n)$$

$$H'(z) = -0.026z^{-2} - 0.104z^{-3} - 0.204z^{-4} + 0.75z^{-5} - 0.204z^{-6} \\ - 0.104z^{-7} - 0.026z^{-8}$$

$$h(n) = h(N-1-n)$$

$$h(0) = h(10) = 0$$

$$h(1) = h(9) = 0$$

$$h(2) = h(8) = -0.026$$

$$h(3) = h(7) = -0.104$$

$$h(4) = h(6) = -0.204$$

$$h(5) = 0.75$$

$$|H(e^{j\omega})| = h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}-n\right) \cos n\omega$$

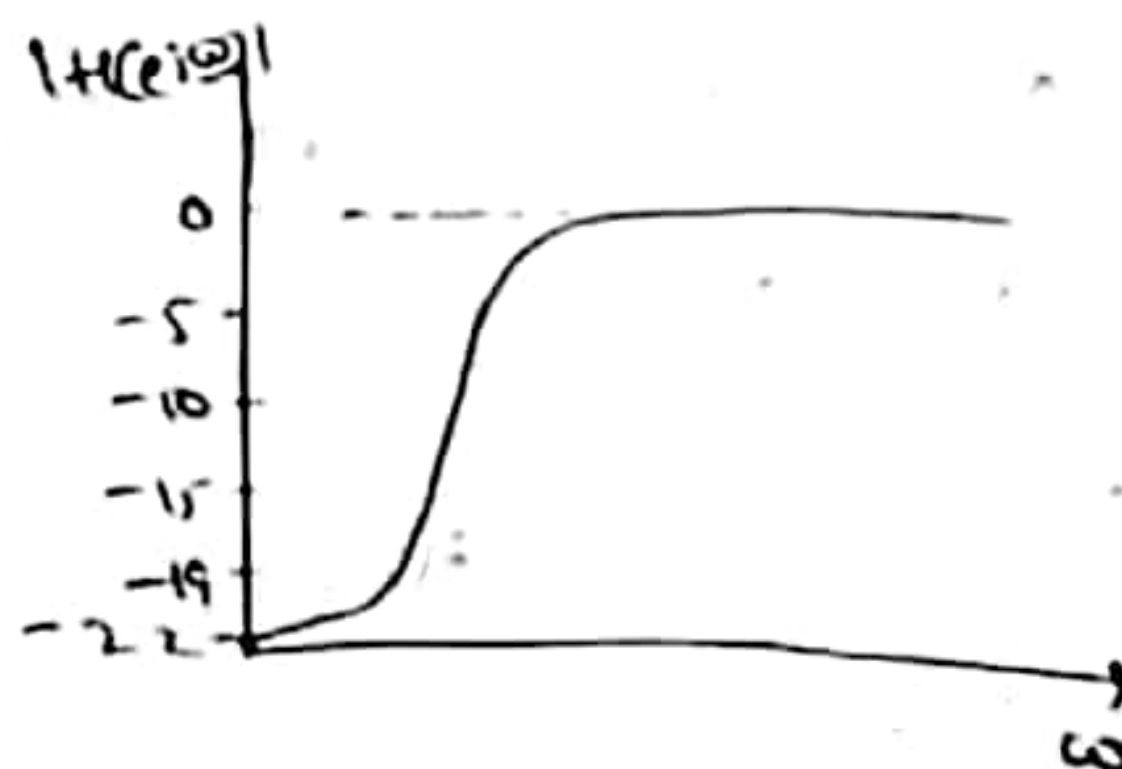
$$= h(5) + \sum_{n=1}^5 2h(5-n) \cos n\omega$$

$$= h(5) + 2h(4) \cos \omega + 2h(3) \cos 2\omega + 2h(2) \cos 3\omega + 2h(1) \cos 4\omega \\ + 2h(0) \cos 5\omega$$

$$= 0.75 + 2(-0.204) \cos \omega + 2(-0.104) \cos 2\omega + 2(-0.026) \cos 3\omega$$

$$|H(e^{j\omega})| = 0.75 - 0.408 \cos \omega - 0.208 \cos 2\omega - 0.052 \cos 3\omega$$

ω	0	15	30	45	60	75	90	105	120	135	150	165	180
$ H(e^{j\omega}) $	0.75	0.672	0.574	0.467	0.355	0.237	0.127	0.0326	0.0037	0.052	0.015	0	0.017



$$h_d(0) = 0.75$$

$$h_d(1) = h_d(-1) = -0.225$$

$$h_d(2) = h_d(-2) = -0.159$$

$$h_d(3) = h_d(-3) = -0.075$$

$$\left\{ \begin{aligned} \omega_h(n) &= 0.54 + 0.46 \cos\left(\frac{n\pi}{4}\right) \\ &= 0.015 \text{ rad/s} \end{aligned} \right.$$

$$h_d(u) = h_d(-u) = 0$$

$$h_d(5) = h_d(-5) = 0.045$$

$$w_h(n) = w_h(-n)$$

$$w_h(0) = 0.54 + 0.46 = 1$$

$$w_h(1) = w_h(-1) = 0.54 + 0.46 \cos(\pi/5) = 0.912$$

$$w_h(2) = w_h(-2) = 0.54 + 0.46 \cos(2\pi/5) = 0.682$$

$$w_h(3) = w_h(-3) = 0.54 + 0.46 \cos(3\pi/5) = 0.398$$

$$w_h(4) = w_h(-4) = 0.54 + 0.46 \cos(4\pi/5) = 0.1678$$

$$w_h(5) = w_h(-5) = 0.54 + 0.46 \cos(\pi) = 0.08$$

The filter coefficients using hamming window sequence are

$$h(n) = h_d(n) w_h(n); -5 \leq n \leq 5$$

0

; otherwise

$$h(0) = h_d(0) w_h(0) = 0.7571 = 0.75$$

$$h(1) = h(-1) = h_d(1) w_h(1) = -0.225 \times 0.912 = -0.2052$$

$$h(-2) = h(2) = h_d(2) w_h(2) = -0.159 \times 0.682 = -0.1084$$

$$h(-3) = h(3) = h_d(3) w_h(3) = -0.075 \times 0.398 = -0.03$$

$$h(-4) = h(4) = h_d(4) w_h(4) = 0$$

$$h(-5) = h(5) = h_d(5) w_h(5) = 0.045 \times 0.08 = 0.0036$$

$$H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n) [z^{-n} + z^{-n}]$$

$$= 0.75 + h(1) [z^{-1} + z^{-1}] + h(2) [z^{-2} + z^{-2}] + h(3) [z^{-3} + z^{-3}]$$

$$+ h(4) [z^{-4} + z^{-4}] + h(5) [z^{-5} + z^{-5}]$$

$$= 0.75 - 0.2052z^{-1} - 0.2052z^{-1} - 0.1084z^{-2} - 0.1084z^{-2} - 0.0323z^{-3}$$

$$+ 0.0036z^{-5} + 0.0036z^{-5}$$

Realizable T.F

$$H'(z) = z^{-\left(\frac{N-1}{2}\right)} H(z) = z^{-5} H(z)$$

$$= 0.75z^{-5} - 0.205z^{-4} - 0.205z^{-6} - 0.108z^{-3} - 0.108z^{-7} \\ + 0.03z^{-2} - 0.03z^{-8} + 0.0036z^0 + 0.0036z^{-10} \\ = 0.0036 - 0.03z^{-2} - 0.108z^{-3} - 0.205z^{-4} + 0.75z^{-5} \\ - 0.205z^{-6} - 0.108z^{-7} - 0.03z^{-8} + 0.0036z^{-10}$$

$$h(n) = h(N-n)$$

$$w_h(n) = w_h(N-n)$$

Caused filter coefficients,

$$h(n) = h(N-1-n)$$

$$h(0) = h(10) = 0.0036$$

$$h(1) = h(9) = 0$$

$$h(2) = h(8) = 0.03$$

$$h(3) = h(7) = -0.108$$

$$h(4) = h(6) = -0.205$$

$$h(5) = 0.75$$

Magnitude fun of $|H(e^{j\omega})| = h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} h\left(\frac{N-1}{2} - n\right) \cos n\omega$

$$= h(5) + \sum_{n=1}^5 2h(5-n) \cos n\omega$$

$$= 0.75 + 2h(4) \cos \omega + 2h(3) \cos 2\omega + 2h(2) \cos 3\omega \\ + 2h(1) \cos 4\omega + 2h(0) \cos 5\omega$$

$$= 0.75 - 0.410 \cos \omega - 0.216 \cos 2\omega - 0.06 \cos 3\omega + 0.0072 \cos 4\omega$$

* The normalized Ideal (desired)-freq. response for filter design using windows.

→ Low pass filter

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega} & ; -\omega_c \leq \omega \leq \omega_c \\ 0 & ; -\pi \leq \omega \leq -\omega_c \\ 0 & ; \omega_c \leq \omega \leq \pi \end{cases}$$

→ High pass filter :-

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega} & ; -\pi \leq \omega \leq -\omega_c \\ e^{-j\omega} & ; \omega_c \leq \omega \leq \pi \\ 0 & ; -\omega_c < \omega < \omega_c \end{cases}$$

→ Band pass filter :-

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega} & ; -\omega_{c2} \leq \omega \leq \omega_{c1} \\ e^{-j\omega} & ; \omega_{c1} \leq \omega \leq \omega_{c2} \\ 0 & ; -\pi \leq \omega < -\omega_{c2} \\ 0 & ; -\omega_{c1} < \omega < \omega_{c1} \\ 0 & ; \omega_{c2} < \omega \leq \pi \end{cases}$$

→ Band stop filter :-

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega} & ; -\pi \leq \omega \leq -\omega_{c2} \\ e^{-j\omega} & ; -\omega_{c1} \leq \omega \leq \omega_{c1} \\ e^{-j\omega} & ; \omega_{c2} < \omega < \pi \\ 0 & ; -\omega_{c2} \leq \omega \leq -\omega_{c1} \\ 0 & ; \omega_{c1} < \omega < \omega_{c2} \end{cases}$$

⑤. Design a HPF using hamming window with a cut off freq. of 1.2 rad/sec and $N=9$.

Sol: The desired freq. response $H_d(e^{j\omega})$ for a HPF is

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\alpha\omega} & ; -\pi \leq \omega \leq \omega_c \\ e^{-j\pi\omega} & ; \omega_c \leq \omega \leq \pi \\ 0 & ; -\omega_c < \omega < \omega_c \end{cases}$$

$$N = 9$$

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{-\omega_c} H_d(e^{j\omega}) e^{j\omega n} d\omega + \frac{1}{2\pi} \int_{\omega_c}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{-\omega_c} e^{-j\alpha\omega} e^{j\omega n} d\omega + \frac{1}{2\pi} \int_{\omega_c}^{\pi} e^{-j\pi\omega} e^{j\omega n} d\omega \end{aligned}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{-\omega_c} e^{j\omega(n-\alpha)} d\omega + \frac{1}{2\pi} \int_{\omega_c}^{\pi} e^{j\omega(n-\pi)} d\omega$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\omega(n-\alpha)}}{n-\alpha} \right]_{-\pi}^{-\omega_c} + \frac{1}{2\pi} \left[\frac{e^{j\omega(n-\pi)}}{n-\pi} \right]_{\omega_c}^{\pi}$$

$$= \frac{1}{2\pi} \frac{e^{-j\omega_c(n-\alpha)} - e^{-j\pi(n-\alpha)}}{j(n-\alpha)} + \frac{1}{2\pi} \frac{e^{j\pi(n-\pi)} - e^{j\omega_c(n-\pi)}}{j(n-\pi)}$$

$$= \frac{1}{2\pi j(n-\alpha)} \left[e^{-j\omega_c(n-\alpha)} - e^{-j\pi(n-\alpha)} \right] + \frac{1}{2\pi j(n-\pi)} \left[e^{j\pi(n-\pi)} - e^{j\omega_c(n-\pi)} \right]$$

$$= \frac{1}{2\pi j(n-\alpha)} \left[-2j \sin \omega_c(n-\alpha) + 2j \sin \pi(n-\alpha) \right]$$

$$h_d(n) = \frac{1}{\pi(n-\alpha)} \left[-\sin \omega_c(n-\alpha) + \sin \pi(n-\alpha) \right] \quad , \quad n \neq \alpha = \frac{N-1}{2}$$

for other values

$n = \alpha$. By using L-Hospital's rule

$$h_d(n) = \lim_{n \rightarrow \alpha} \frac{1}{\pi(n-\alpha)} \left[\sin \omega_c(n-\alpha) + \sin \pi(n-\alpha) \right]$$

$$= \lim_{n \rightarrow \infty} \left[\frac{\sin \pi(n-\alpha)}{\pi(n-\alpha)} - \lim_{n \rightarrow \infty} \frac{\sin \pi(n-\alpha)}{\pi(n-\alpha)} \right]$$

$$h_d(n) = 1 - \frac{\omega_c}{\pi} \quad n = \alpha = \frac{N-1}{2} = \frac{9-1}{2} = 4.$$

⑥ Design a band stop filter to reject freq. in the range 1 to 2 rad/sec. using rectangular window with $N=7$.

Sol. $\omega_{c1} = 1 \text{ rad/sec}$, $\omega_{c2} = 2 \text{ rad/sec}$.

$$\alpha = \frac{N-1}{2} = 3$$

The desired freq. response for BSF.

$$H(e^{j\omega}) = \begin{cases} e^{-j\alpha\omega} & ; -\pi \leq \omega \leq -\omega_{c2}, -\omega_{c1} \leq \omega \leq \omega_{c1}, \omega_{c2} \leq \omega \leq \pi \\ 0 & ; \text{other values} \end{cases}$$

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega.$$

$$= \frac{1}{2\pi} \int_{-\pi}^{-\omega_{c2}} e^{-j\alpha\omega} e^{j\omega n} d\omega + \frac{1}{2\pi} \int_{-\omega_{c1}}^{\omega_{c1}} e^{-j\alpha\omega} e^{j\omega n} d\omega + \frac{1}{2\pi} \int_{\omega_{c2}}^{\pi} e^{-j\alpha\omega} e^{j\omega n} d\omega$$

$$= \frac{1}{\pi(n-\alpha)} \left[\sin(n-\alpha)\omega_{c1} + \sin(n-\alpha)\pi - \sin(n-\alpha)\omega_{c2} \right].$$

At $n = \alpha$.

$$h_d(n) = 1 + \frac{\omega_{c1}}{\pi} - \frac{\omega_{c2}}{\pi} \quad \text{for } h_d(3)$$

⑦ Design a Band pass FIR filter for the following specification Cut off freq. = 400 Hz & 800 Hz. Sample freq. 2000 Hz. $N=11$.

Sol. Cut off freq. $f_{c1} = 400 \text{ Hz}$
 $f_{c2} = 800 \text{ Hz}$
 $f_s = 2000 \text{ Hz}$.

$$\text{Normalized cut off freq. } \omega_{c1} = \frac{2\pi f_{c1}}{f_s} = \frac{2\pi \times 400}{2000}$$

$$\omega_c = \frac{2\pi(800)}{2000} = \frac{4\pi}{5}$$

Frequency Sampling method

→ Design of FIR filter by freq. sampling method.

Procedure for Type-I

→ Choose the ideal (desired) freq. response $H_d(e^{j\omega})$

→ Sample $H_d(e^{j\omega})$ at N points by taking $\omega = \omega_k = \frac{2\pi k}{N}$

where $k = 0, 1, 2, \dots, N-1$ to generate the sequence

→ $\tilde{H}(k)$ (Tild).

$$\therefore \tilde{H}(k) = \underset{H_d(\omega)}{H_d(e^{j\omega})} \Big|_{\omega = \frac{2\pi k}{N}}$$

→ Compute the N -sample of $h(n)$ using the following eqn.

Case I: N is odd.

$$h(n) = \frac{1}{N} \left[\tilde{H}(0) + 2 \sum_{k=1}^{\frac{N-1}{2}} \text{Re} \left[\tilde{H}(k) e^{j \frac{2\pi nk}{N}} \right] \right]$$

Case II: N is even

$$h(n) = \frac{1}{N} \left[\tilde{H}(0) + 2 \sum_{k=0}^{\frac{N}{2}-1} \text{Re} \left[\tilde{H}(k) e^{j \frac{2\pi nk}{N}} \right] \right]$$

→ Take z.T of the impulse response of $h(n)$

to get the T.F $H(z)$

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

$$\tilde{H}(k) = |H(k)| = |H(k)| e^{j0}$$

→ for Linear phase filter $\Theta = -\alpha\omega$

$$\Theta = -\left(\frac{N-1}{2}\right) \frac{2\pi k}{N}$$

$$\Theta = -\left(\frac{N-1}{N}\right) \pi k ; \quad k = 0, 1, \dots, N-1$$

The filter coefficients $h(n)$ can be obtained by finding IDFT of $H(k)$

$$h(n) = \frac{1}{N} \sum_{k=0}^{N-1} H(k) e^{j \frac{2\pi kn}{N}} ; n=0,1,2,\dots,N-1$$

If $h(n)$ the impulse response of the filter is to be real valued s.t. the freq. samples $H(k)$ must satisfy the symmetry requirement for N odd or even

$$H(N-k) = H^*(k) ; k=0,1,2,\dots,N-1$$

In addition for N even $h(\frac{N}{2})=0$.

With the freq. response $H(k)$ the magnitude response is an even function.

$$|H(k)| = |H(N-k)| ; k=0,1,\dots,N-1$$

The phase is an odd function: we represent phase fun

$$\theta(k) = -\theta(N-k)$$

$$\theta(N-k) = -\left(\frac{N-1}{N}\right)\pi(N-k)$$

$$= -\left(\frac{N-1}{N}\right)(\pi N - \pi k)$$

$$= -\left(\frac{N-1}{N}\right)\pi N + \left(\frac{N-1}{N}\right)\pi k$$

$$= -(N-1)\pi + \left(\frac{N-1}{N}\right)\pi k$$

$\theta(k)$ for N odd is given by $\theta(k)$

$$\theta(k) = -\left(\frac{N-1}{N}\right)\pi k ; k=0,1,2,\dots,N-1$$

$$= (N-1)\pi - \left(\frac{N-1}{N}\right)\pi k ; \text{for } k = \frac{N+1}{2}, \dots, N$$

$\theta(k)$ for N even is given by

$$\theta(k) = -\left(\frac{N-1}{N}\right)\pi k ; k=0,1,\dots,\frac{N}{2}-1$$

$$= 0 \quad ; \quad k = \frac{N}{2}$$

$$= (N-1)\pi - \left(\frac{N-1}{N}\right)\pi k \quad ; \quad k = \frac{N}{2} + 1, \dots, N-1$$

$$\theta(k) = -\theta(N-1-k)$$

for N odd

$$H(k) = |H(k)| e^{-j \left(\frac{N-1}{N}\right)\pi k} \quad \text{for } k = 0, 1, \dots, \frac{N-1}{2}$$

$$= |H(k)| e^{+j \left[(N-1)\pi - \left(\frac{N-1}{N}\right)\pi k \right]} \quad \text{for } k = \frac{N+1}{2}, \dots, N-1$$

for N even

$$H(k) = |H(k)| e^{-j \left(\frac{N-1}{N}\right)\pi k} \quad ; \quad k = 0, 1, \dots, \frac{N}{2} - 1$$

$$= |H(k)| \quad ; \quad k = \frac{N}{2}$$

$$= |H(k)| e^{+j \left[(N-1)\pi - \left(\frac{N-1}{N}\right)\pi k \right]} \quad ; \quad k = \frac{N}{2} + 1, \dots, N-1$$

for N odd

$$h(n) = \frac{1}{N} \sum_{k=0}^{N-1} \left[\tilde{H}(0) + 2 \sum_{k=1}^{\frac{N-1}{2}} \operatorname{Re} \left[H(k) e^{j \frac{2\pi n k}{N}} \right] \right]$$

for N even

$$h(n) = \frac{1}{N} \left[\tilde{H}(0) + 2 \sum_{k=1}^{\frac{N}{2}-1} \operatorname{Re} \left[H(k) e^{j \frac{2\pi n k}{N}} \right] \right]$$

Once the filter coefficient $h(n)$ have been determined

the dy fin filter is given by

$$\Rightarrow H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

D Determine the filter coefficients $h(n)$ obtained by sampling $H(\omega) = H(e^{j\omega}) = e^{-j(N-1)\omega/2}$; $0 \leq |\omega| < \pi$
for $N=7$. $= 0$; $\pi/2 \leq |\omega| < \pi$

Sol: $\omega_k = \frac{2\pi}{N} k$, $k=0, 1, \dots, N-1$.

$H(k) = H(\omega_k) = |H(k)| e^{j\theta(k)}$

$\omega_k = \frac{2\pi}{N} k$ $k=0, 1, 2, 3, 4, 5, 6$

$\omega_0 = 0$, $k=0$ $\frac{2}{7} = 0.28$

$\omega_1 = \frac{2\pi}{7}$, $k=1$ $\frac{4}{7} = 0.57$

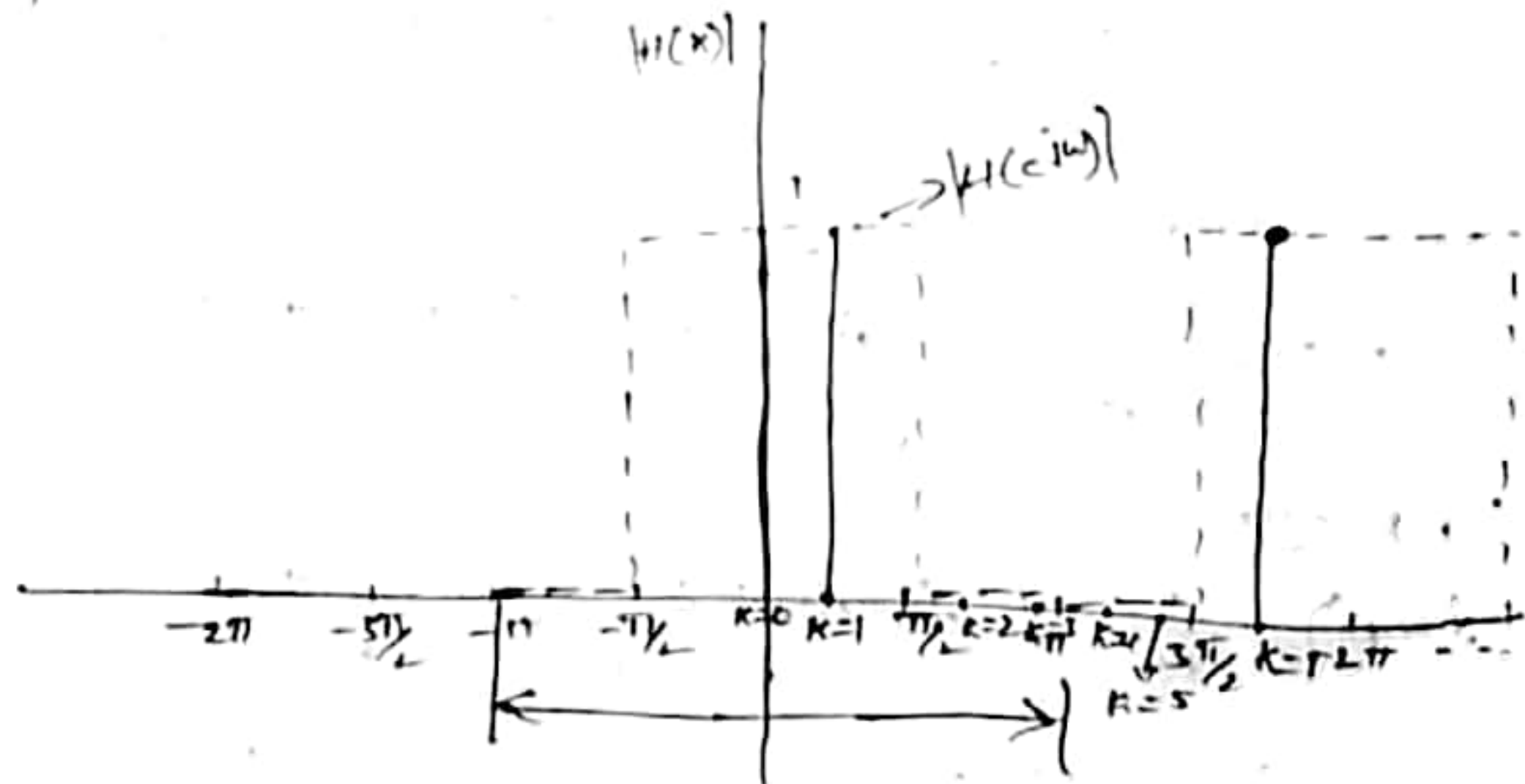
$\omega_2 = \frac{4\pi}{7}$, $k=2$ $\frac{6}{7} = 0.857$

$\omega_3 = \frac{6\pi}{7}$, $k=3$ $\frac{8}{7} = 1.14$

$\omega_4 = \frac{8\pi}{7}$, $k=4$ $\frac{10}{7} = 1.42$

$\omega_5 = \frac{10\pi}{7}$, $k=5$ $\frac{12}{7} = 1.71$

$\omega_6 = \frac{12\pi}{7}$, $k=6$



$|H(k)| = 1$; $k=0, 1, 2, 3, 4, 5$

$= 0$; $k=6$

$\theta(k) = -\left(\frac{N-1}{N}\right)\pi k$; $k=0, 1, 2, 3, 4, 5$

$= -\frac{6}{7}\pi k$; $k=0, 1, 2, 3, 4, 5$

odd

$\theta(k) = (N-1)\pi - \left(\frac{N-1}{N}\right)\pi k$

$= 6\pi - \frac{6\pi}{7} k$; $k=4, 5, 6$

$$H(k) = |H(k)| e^{j\theta(k)}$$

$$H(k) = 1 e^{-j\frac{6}{7}\pi k} ; k=0,1$$

$$0, k=2,3,4,5$$

$$= 1 e^{j\left[6\pi - \frac{6}{7}\pi k\right]} ; k=6$$

N odd

$$h(n) = \frac{1}{N} \left[H(0) + 2 \sum_{k=1}^{\frac{N-1}{2}} \text{Re} \left[H(k) e^{j\frac{2\pi nk}{N}} \right] \right]$$

$$= \frac{1}{7} \left[1 + 2 \sum_{k=1}^3 \text{Re} \left[H(k) e^{j\frac{2\pi nk}{7}} \right] \right]$$

$$= \frac{1}{7} \left[1 + 2 \text{Re} \left[e^{-j\frac{6}{7}\pi(1)} e^{j\frac{2\pi n(1)}{7}} \right] \right]$$

$$h(n) = \frac{1}{7} \left[1 + 2 \text{Re} \left[e^{-j\frac{2\pi}{7}(n-3)} \right] \right]$$

$$= \frac{1}{7} \left[1 + 2 \cos \frac{2\pi}{7} (n-3) \right]$$

$$h(n) = h(6-1-n)$$

$$h(n) = h(6-n)$$

$$h(0) = h(6) = \frac{1}{7} \left[1 + 2 \cos \frac{6\pi}{7} \right] = -0.11456$$

$$h(1) = h(5) = \frac{1}{7} \left[1 + 2 \cos \frac{4\pi}{7} \right] = 0.099$$

$$h(2) = h(4) = \frac{1}{7} \left[1 + 2 \cos \frac{2\pi}{7} \right] = 0.321$$

$$h(3) = h(3) = \frac{1}{7} \left[1 + 2 \cos(0) \right] = 0.4285$$

Using freq. Sampling method Design a BPF with the following specification.

→ Sampling freq $f_s = 8000 \text{ Hz}$

→ Cut off freq $f_{c1} = 1000 \text{ Hz}$, $f_{c2} = 3000 \text{ Hz}$

Determine the filter coefficient $N=7$.

$$\omega_k = \frac{2\pi}{N} k$$

$$\omega_0 = 0, k=0 \quad \omega_6 = \frac{12}{7}\pi \quad k=6$$

$$\omega_1 = \frac{2}{7}\pi, k=1$$

$$\omega_2 = \frac{4}{7}\pi, k=2$$

$$\omega_3 = \frac{6}{7}\pi, k=3$$

$$\omega_4 = \frac{8}{7}\pi, k=4$$

$$\omega_5 = \frac{10}{7}\pi, k=5$$

$$\omega_{c1} = \frac{2\pi f_{c1}}{f_s} = \frac{2\pi 1000}{8000} = \frac{\pi}{4}$$

$$\omega_{c2} = \frac{2\pi 3000}{8000} = \frac{3\pi}{4}$$

$$|H(k)| = 1; k=1, 2, 5, 6$$

$$0; k=0, 3, 4$$

$$\theta(k) = -\left(\frac{N-1}{N}\right)\pi k; k=0, 1, 2, 3$$

$$= -\frac{6}{7}\pi k; k=0, 1, 2, 3$$

$$\theta(k) = (N-1)\pi - \left(\frac{N-1}{N}\right)\pi k; k=4, 5, 6$$

$$= 6\pi - \frac{6}{7}\pi k; k=4, 5, 6$$

$$H(k) = |H(k)| e^{j\theta(k)}$$

$$H(k) = 0; k=0, 3, 4$$

$$= 1 e^{-j\frac{6}{7}\pi k}; k=1, 2$$

$$= 1 e^{j(6\pi - \frac{6}{7}\pi k)}; k=5, 6$$

$$h(n) = \frac{1}{N} \left[H(0) + 2 \sum_{k=1}^{\frac{N-1}{2}} \operatorname{Re} \left[H(k) e^{j\frac{2\pi n k}{N}} \right] \right]$$

$$= \frac{1}{7} \left[0 + 2 \sum_{k=1}^3 \operatorname{Re} \left[H(k) e^{j\frac{2\pi n k}{7}} \right] \right]$$

$$= \frac{1}{7} \left[-\operatorname{Re} \left[e^{-j\frac{6}{7}\pi} e^{j\frac{2\pi n}{7}} \right] + 2 \operatorname{Re} \left[e^{-j\frac{12}{7}\pi} e^{j\frac{4\pi n}{7}} \right] \right]$$

$$= \frac{1}{7} \left(2 \operatorname{Re} \left[e^{j \frac{2\pi}{7} (n-3)} \right] + 2 \operatorname{Re} \left[e^{j \frac{4\pi}{7} (n-3)} \right] \right)$$

$$h(n) = \frac{1}{7} \left(2 \cos \frac{2\pi}{7} (n-3) + 2 \cos \frac{4\pi}{7} (n-3) \right)$$

$$h(n) = h(N-1-n)$$

$$h(0) = h(6) = \frac{1}{7} \left(2 \cos \frac{6\pi}{7} + 2 \cos \frac{12\pi}{7} \right) = -0.0992$$

$$h(1) = h(5) = \frac{1}{7} \left(2 \cos \frac{5\pi}{7} + 2 \cos \frac{10\pi}{7} \right) = -0.321$$

$$h(2) = h(4) = \frac{1}{7} \left(2 \cos \frac{4\pi}{7} + 2 \cos \frac{8\pi}{7} \right) = 0.1145$$

$$h(3) = \frac{1}{7} \left(2 \cos 0 + 2 \cos 0 \right) = \frac{4}{7} = 0.57$$

Structures for realization of FIR filter ^{Sly's}

The diff. types of structures for realizing the FIR

Sly's 1

1. Direct form realization.
2. Transposed form realization.
3. Cascade realization
4. Lattice structure realization.
5. Linear phase realization

The eq of FIR sys is

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{N-1} b_k z^{-k}$$

$$= b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_{N-1} z^{-(N-1)}$$

$$H(z) = z^{-1} [h(n)] = \sum_{n=0}^{N-1} h(n) z^{-n}$$

Change index in to k

$$= \sum_{k=0}^{N-1} h(k) z^{-k} = h(0) + h(1) z^{-1} + \dots + h(N-1) z^{-(N-1)}$$

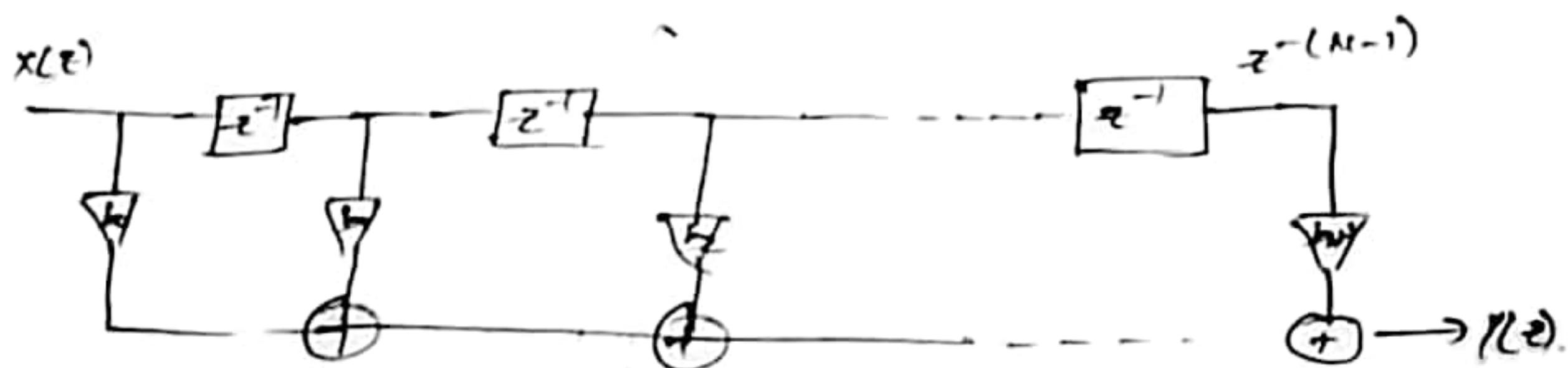
$$b_0 = h(0), b_1 = h(1), \dots, b_k = h(k)$$

Direct form realization of FIR sys:-

The direct form structure can be obtained from the general eq'n $Y(z)$ of the FIR sys.

$$Y(z) = \sum_{k=0}^{N-1} b_k z^{-k} X(z)$$

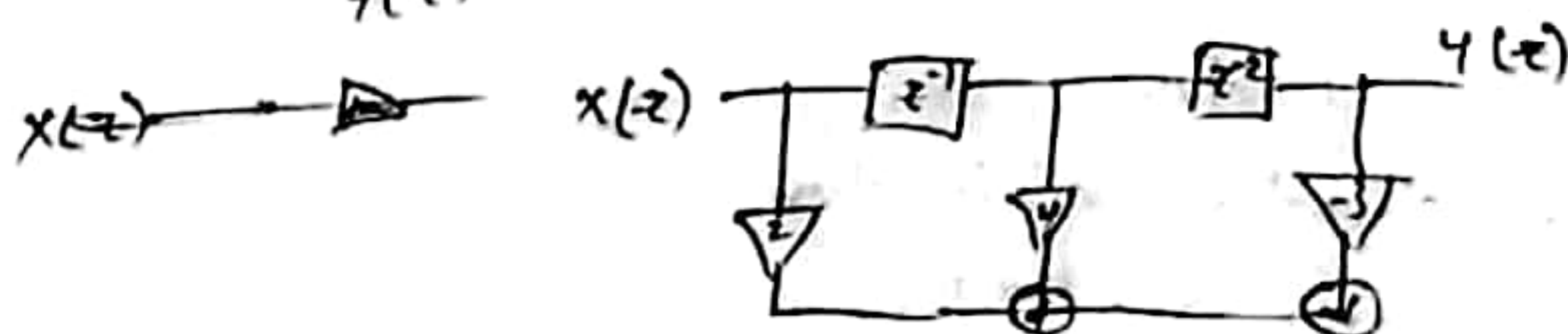
$$= b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + \dots + b_{N-1} z^{-(N-1)} X(z)$$

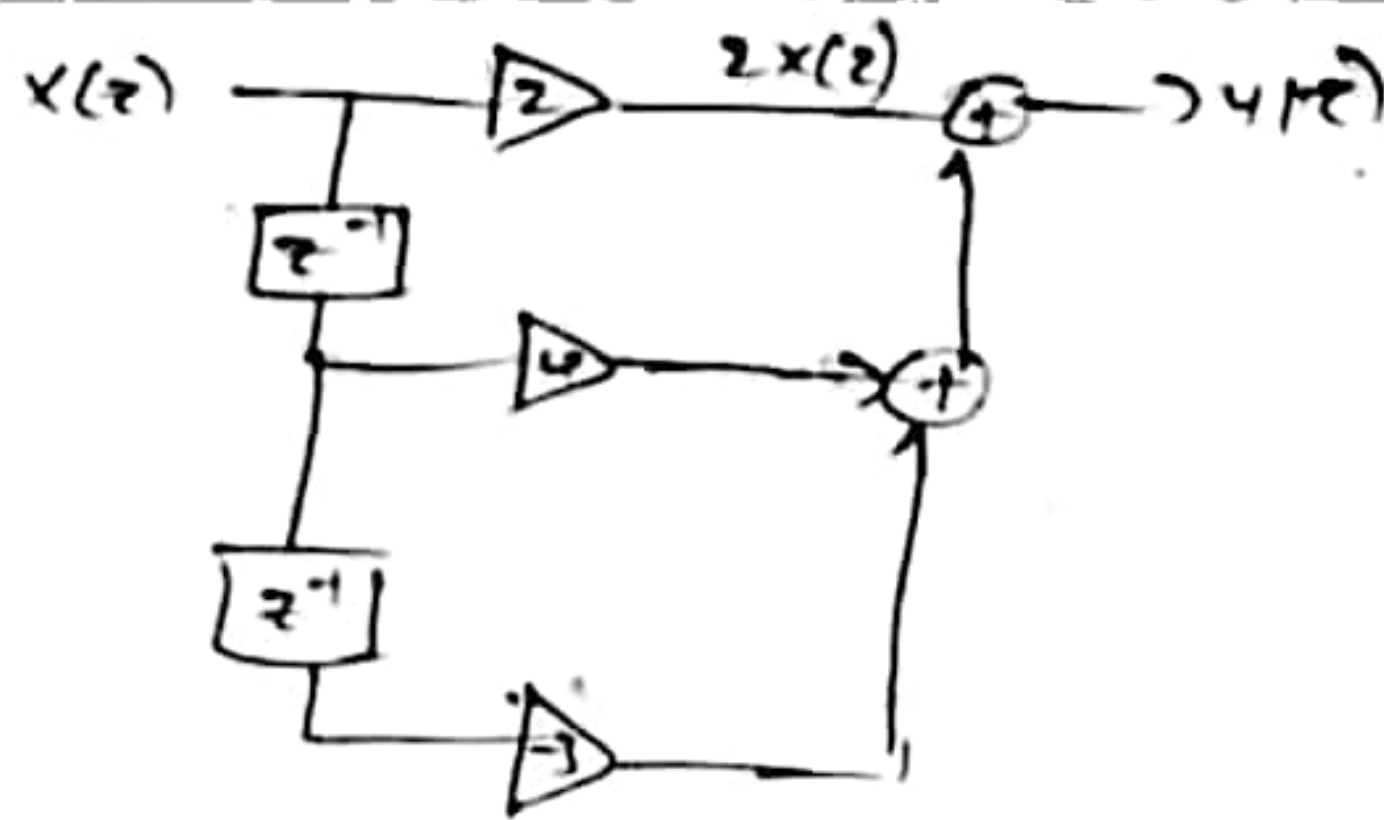


Q. Realize the second order FIR sys $y(n) = 2x(n) + 4x(n-1) - 3x(n-2)$ using direct form structure.

sol:- Given $y(n) = 2x(n) + 4x(n-1) - 3x(n-2)$

$$Y(z) = 2X(z) + 4z^{-1}X(z) - 3z^{-2}X(z)$$





Lattice structure Realization of FIR systems:-

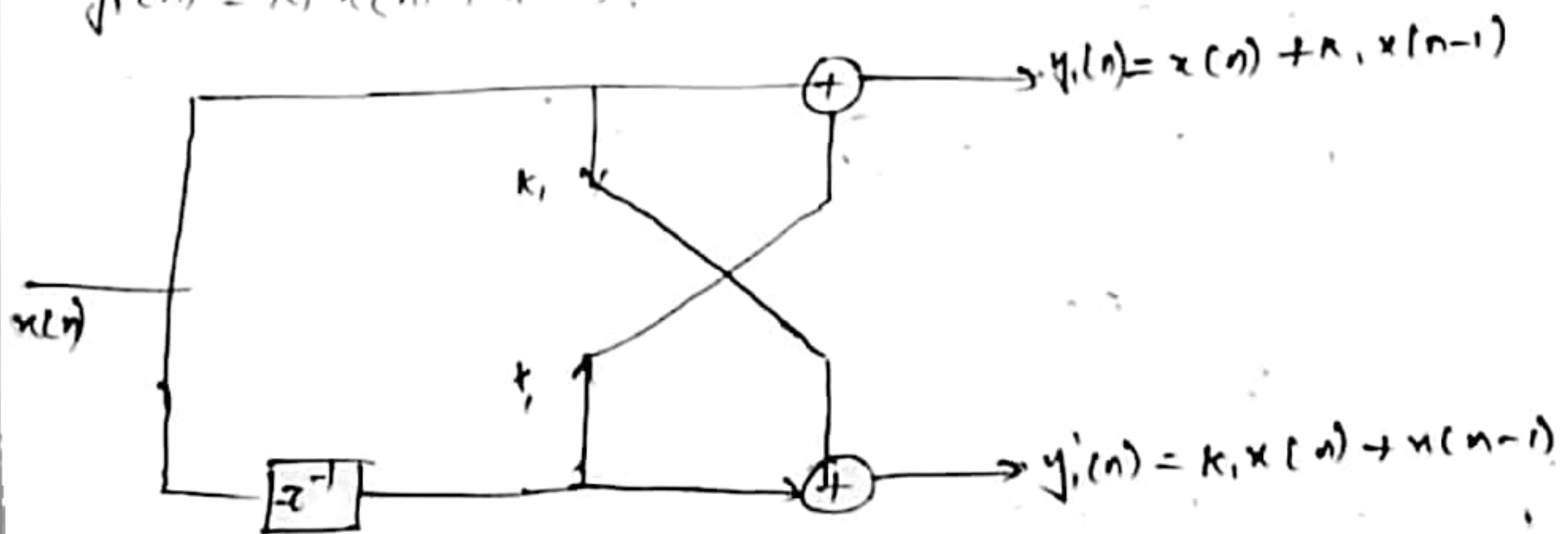
The lattice structure consists of two diff. paths through which the i/p $x(n)$ is processed. Hence the lattice structure has two diff. output setups $y(n)$ & $y'(n)$.

$y(n)$ is the real o/p & $y'(n)$ is the supporting o/p, which offers support for obtaining the o/p for the next stage.

→ A single stage lattice structure shown in the following fig. where k is reflection coefficient.

$$y_1(n) = x(n) + k_1 x(n-1)$$

$$y'_1(n) = k_1 x(n) + x(n-1)$$



(P). Realize the S(y)

Procedure to realize the Lattice structure of the FIR S(y):-

→ If the Coeff of the present i/p $x(n)$ is not unity. Convert it to unity by taking common of the Coefficient of the present i/p

→ Find the order of the diff. eq'n and compare the Coeff. of the given diff eq'n with the Coeff. of the same order Lattice structure o/p involving the reflection Coeff. $k_1, k_2, k_3, \dots$

→ Assign the Calculated values of $k_1, k_2, \dots$ and compare the structure.

(P). Realize a sly. with $G + H(z) = 5 + 3z^{-1}$ by using Lattice structure.

Sol:- Given $H(z) = 5 + 3z^{-1}$

$$\frac{Y(z)}{X(z)} = 5 + 3z^{-1}$$

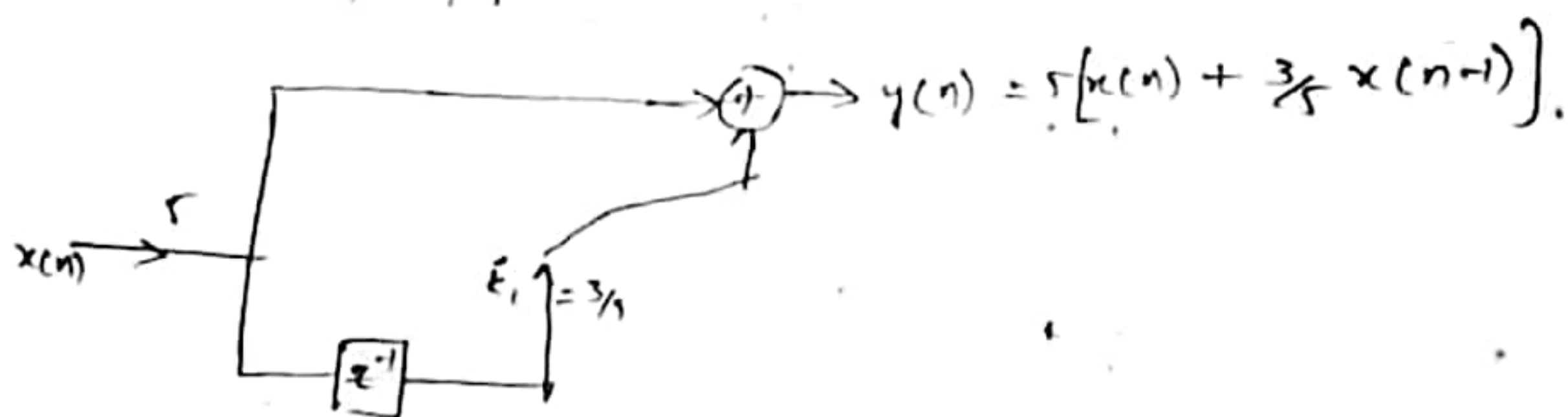
$$Y(z) = 5X(z) + 3z^{-1}X(z)$$

$$y(n) = 5x(n) + 3x(n-1)$$

$$= 5 \left[x(n) + \frac{3}{5} x(n-1) \right]$$

$$y_1(n) = x(n) + k_1 x(n-1)$$

$$k_1 = \frac{3}{5}$$



Two stage Lattice structure :-

The output from the second stage of the Lattice structure which is the second order FIR sly is given as follows.

$$y_2(n) = y_1(n) + k_2 y_1'(n-1)$$

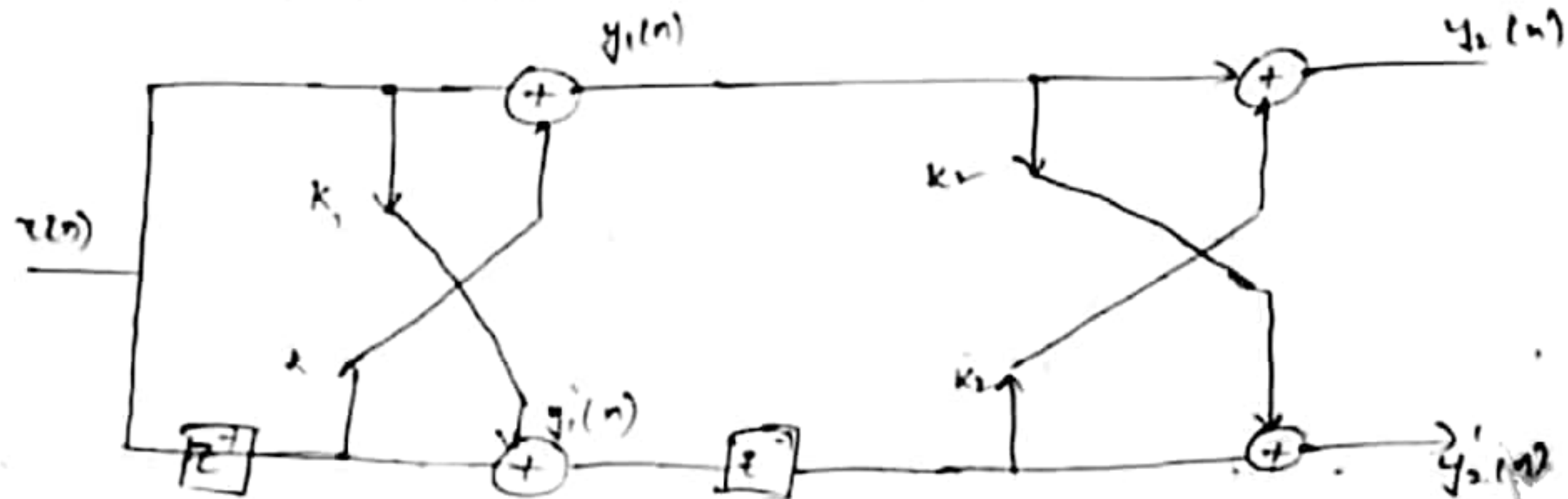
$$y_2'(n) = k_2 y_1(n) + y_1'(n-1)$$

$$y_2(n) = [x(n) + k_1 x(n-1)] + k_2 [k_1 x(n-1) + x(n-2)]$$

$$= x(n) + k_1 x(n-1) [1 + k_2] + k_2 x(n-2)$$

$$y_2'(n) = k_2 [x(n) + k_1 x(n-1)] + [k_1 x(n-1) + x(n-2)]$$

$$= k_2 x(n) + k_1 x(n-1) [1 + k_2] + x(n-2)$$



① Determine the lattice coeff. corresponding to the FIR
 sly with the sly $h(z) = 1 + \frac{7}{9}z^{-1} + \frac{3}{5}z^{-2}$ and
 realize it.

Sol:- $h(z) = 1 + \frac{7}{9}z^{-1} + \frac{3}{5}z^{-2}$

$$y(z) = x(z) + \frac{7}{9}z^{-1}x(z) + \frac{3}{5}z^{-2}x(z)$$

$$y(n) = x(n) + \frac{7}{9}x(n-1) + \frac{3}{5}x(n-2) \quad \text{--- (2)}$$

6/8/19

UNIT-VI

Introduction To programmable DSP

Architecture features of DSP processors:

- 1) DSP processor should have multiple Register so that data can be exchanged from Register to Register fast.
- 2) DSP operations required multiple operands simultaneously. Hence DSP processor should have multiple operand fetching capacitor.
- 3) DSP should have buffers to support the shifting operations.
- 4) The DSP processor should be able to perform multiple and accumulate operations very fast.
- 5) DSP processor should have multiple pointers to support multiple operands, jumps and shift operations.
- 6) DSP processor can be used with generator processor. They should have multi processing capacity.
- 7) To support DSP operations DSP processor should have on-chip memory.
- 8) For real-time applications interrupts and timers are required. Hence DSP processor should have powerful interrupts and timers.

MAC (Multiply and Accumulator):

Most of the operations in DSP involve array multiplication. The operations such as convolution, correlation require multiply and accumulate operations.

In real-time applications the array multiplication and accumulation must be completed before next

sample of i/p comes.
this requires very fast implementation of multiplication and accumulation.

for this purpose the dedicated ALU unit MAC is used is called multiplier and accumulator.

The complete MAC operation is executed in one clock cycle.

In the DSP processor the o/p of the multiplier is store into the product register. this product register contains are added to the Accumulator Register the central ALU.

The DSP process have the special instruction called MACB. i.e; multiply and accumulate with data shift.

MAM (Multiple Accesses Memory)

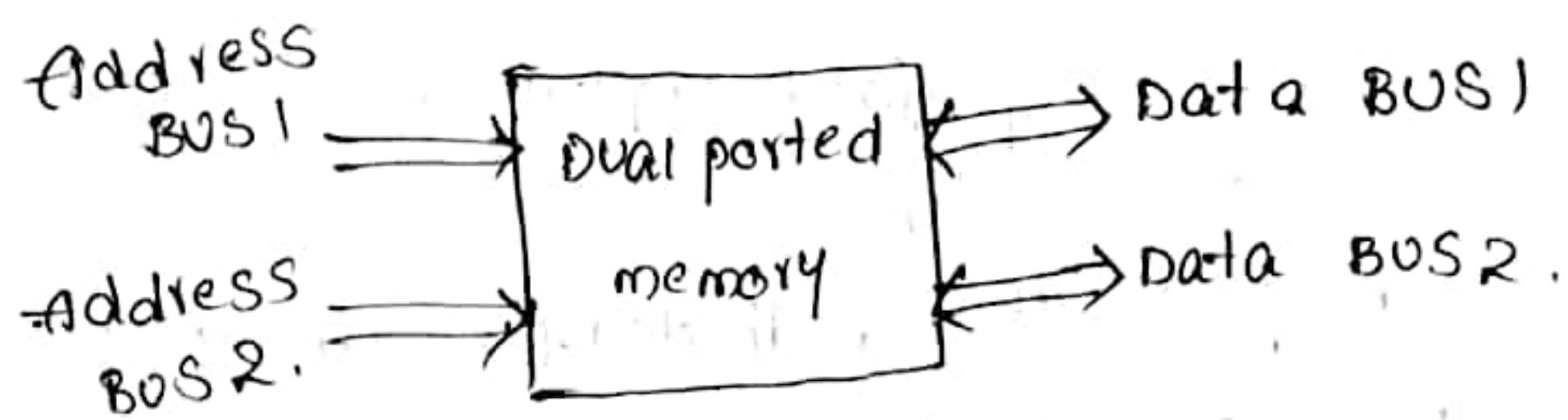
The MAM allows more than one memory accesses in a single clock cycle. the Dual Accesses RAM (DA-RAM) allows 2-memory accesses in a single clock cycle.

The Dual Accesses RAM is connected to the DSP Processes with 2 Address with 2 data buses independently. this gives 4 memory accesses in a single clock period.

MPM (Multi ported Memory)

The MPM has the facility of interfacing multiple address and data buses. the following fig shows the Dual ported memory. this memory has 2 Address buses and 2 data buses separately interfaced.

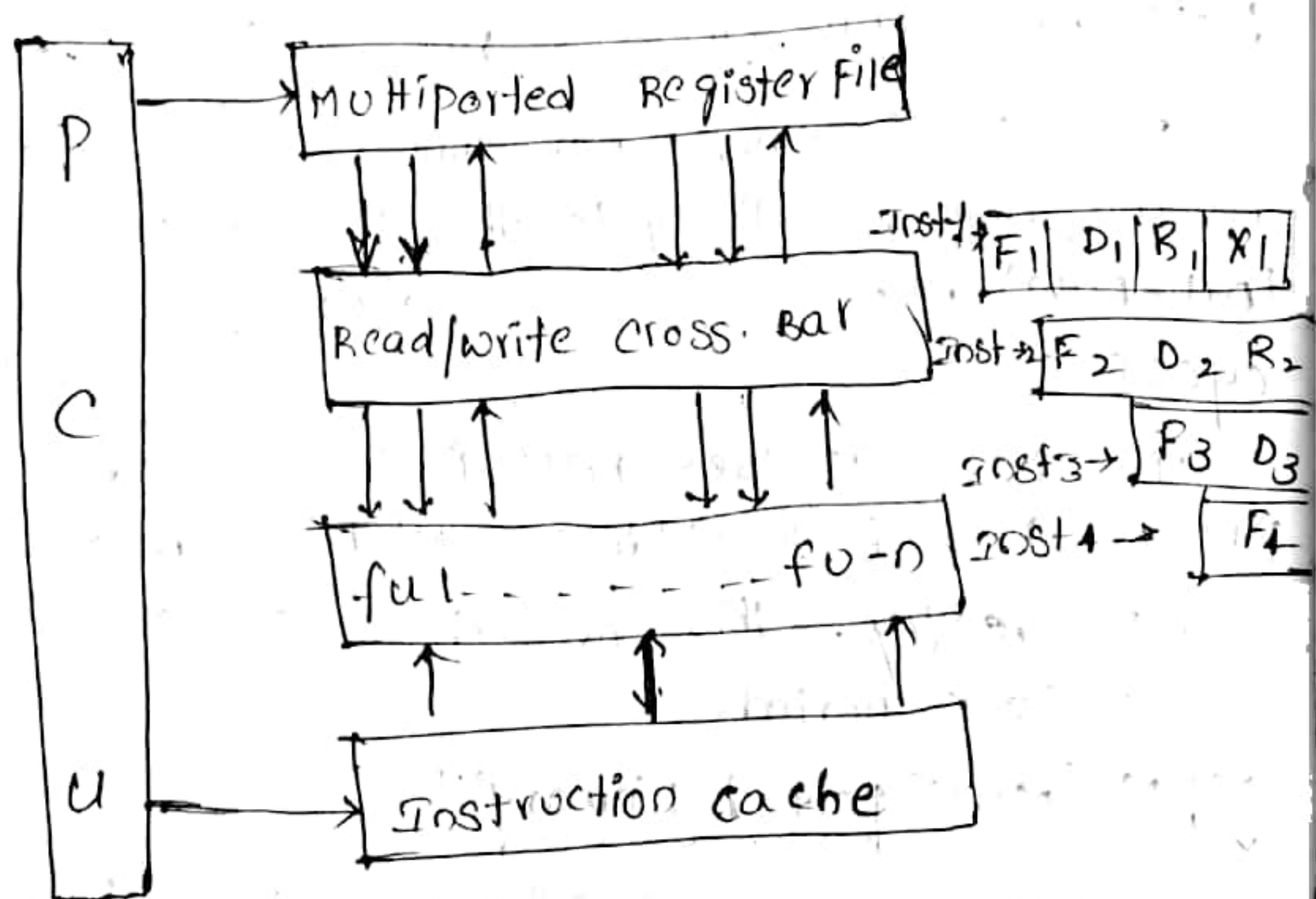
The dual ported memory can allows 2 memory accesses in a single clock period.



With the help of dual port memory the program and data can be store in a single memory chip and they can be accessed simultaneously.

The multi ported memories having increased no. of pins larger chip area which makes it more expensive and large in size.

VLSI (very long instruction word).



PCU → program control unit.

fu → function unit.

Sum of the DSP process use VLSI architecture. Such architecture consists of multiple no. of ALU's, MAP units and shifter registers.

The following fig shows the block diag of VLSI architecture.

The above Architecture consist of multi ported register file. It is used for fetching the operands and storing the results.

The Read/Write cross bar provides parallel Random Accesses by functional units to the multiported register file.

The function units work concurrently with the load (or) store operation of the data b/w REM and Register file.

The program control unit provides the algorithm that executes independent 11^{th} operations.

The performance of VLSI Architecture depends on degree of parallelism.

Normally 8-functional units are preferred this number is limited by H/W cost of the multiported Register file and Read/Write cross bar.

Pipelining:

without pipelining

Value of T	Fetch	Decode	Read	execute.
1	I_1			
2	I_2	I_1		
3		I_2	I_1	
4			I_2	I_1
5	I_2			I_2
6		I_2		
7			I_2	
8				I_2

With pipelining:

Value of T	fetch	decode	Read	execute.
1	I_1			
2	I_2	I_1		
3	I_3	I_2	I_1	
4	I_4	I_3	I_2	I_1
5	I_5	I_4	I_3	I_2
6		I_5	I_4	I_3
7			I_5	I_4
8				I_5

9/2/19.

Any instruction cycle can be split into the following instruction.

- 1) fetch: In this phase an instruction is fetch from the memory.
- 2) decode: In this phase an instruction is decoded.
- 3) Read: An operand required for this instruction is fetch from the data memory.
- 4) execute: The operation is executed and results are stored at appropriate place.

Each of the above instructions can be separately executed in different functional units. to the following fig shows 1) how the instruction is executed without pipeline.

2) from this fig. we can observe that when the instruction I_1 is in fetch, the other units such as

decode, Read and Execute are ideal. This means each functional unit is busy only for 25% of the total time.

from the fig (2) shows the instruction execution with pipeline,

Here we can observe that instruction-1 is in decode phase. next instruction I_2 is fetched.

Similarly, when I_2 goes to decode phase, next instruction I_3 is fetched.

$\therefore$ we observed that all the functional units are executed 4 successive instructions at any time.

compare above two figs we can observed that 5 instructions are executed the same time pipelining misused.

*Special Addressing modes:

The conventional microprocess have addressing modes such as direct, indirect, immediate Addressing modes. The DSP process have additional Addressing modes because of fetch execution is fast.

1) Short immediate Addressing mode:

The operand is specified using a short constant this short constant becomes a part of a single word instruction.

In DSP process 8-bit operand can be specified as one of the operand in single word instructions such as add, subtract, Div, AND, OR, NAND. etc.

2) Short direct Addressing mode:

The lower order Address of the operand is specified in the single word instruction in DSP process the lower 7-bits of the Address are specified as a part of instruction.

Higher 9-bits of the Address are stored the data

Phase point.

3) Memory Map Addressing mode:

The CPU and I/O registers are accessed are memory location this registers are mapped in the starting phase (or) ending phase of the memory phase.

The phase-0 corresponds to the starting page of the memory space.

4) Indirect Addressing mode:

The addressing of the operands are stored in the Indirect Address registers.

In DSP process such registers are called Auxiliary registers.

Any of these auxiliary registers can be updated in the operands fetched by these registers are being executed.

The auxiliary registers are incremented (or) decremented automatically by the value specified. In offset registers. these offset registers called index registers.

5) Bit-Reversal Addressing modes:

For the calculation of FFT the I/P data is required in bit-reversal order. There is no need to re-shuffle

the data in bit reversal sequence.

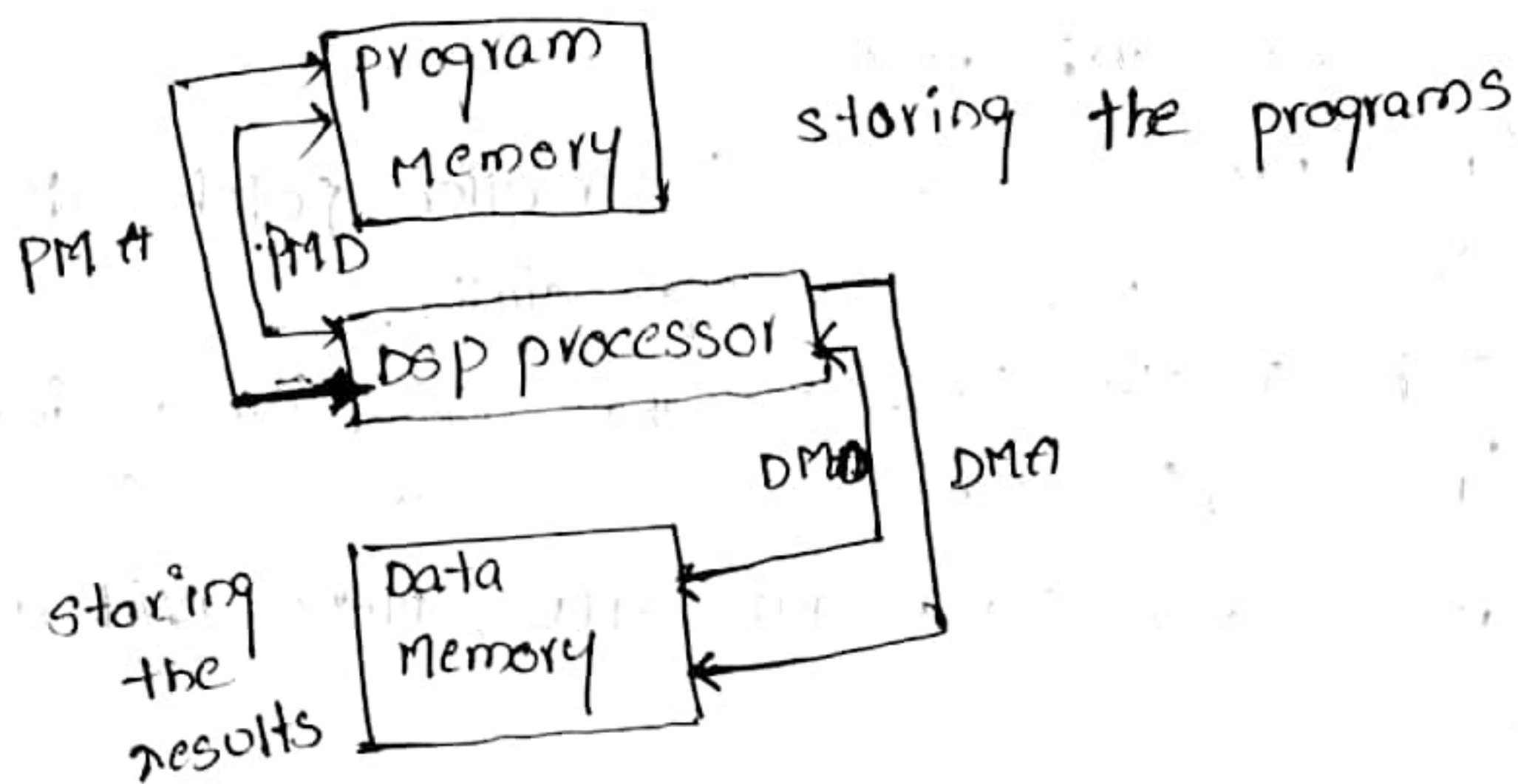
The serially arranged the data in the memory given to the processor in bit reversal mode, with the help of Bit-reversal addressing mode.

6) Circular Addressing mode:

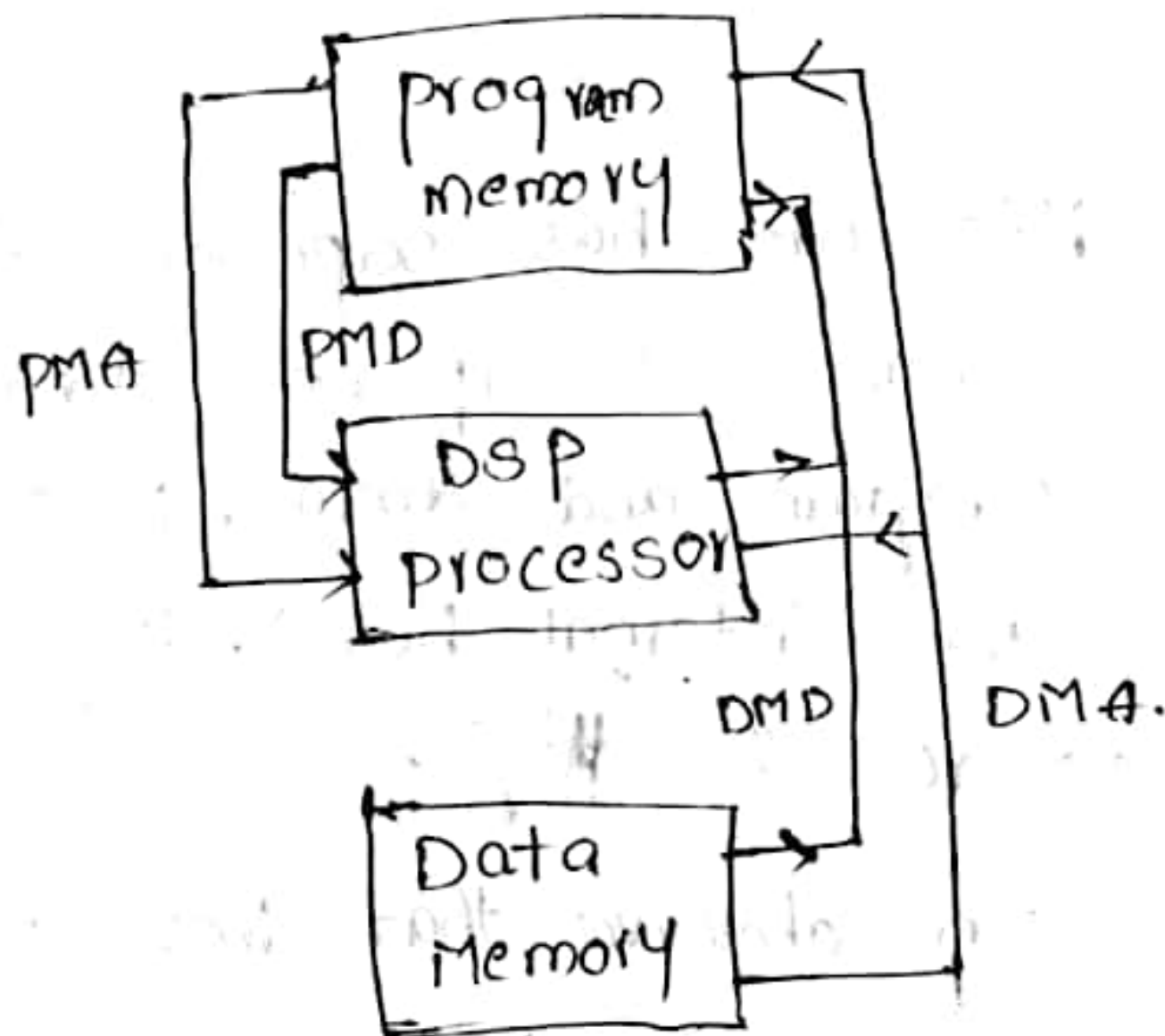
If this mode the data stored in the memory can be read (or) write in circular fashion.

The memory is organized as a circular buffer the beginning and ending addresses of the circular

buffer are continuously monitor
 12/3/19 *** Harvard Architecture:-



Modified Harvard Architecture:-



MACD instruction performs multiply and accumulate in the memory accesses are required.

- 1) Fetch MACD instruction from program memory.
- 2) fetch one of the operands. from program memory.
- 3) fetch second operand from data memory.
- 4) data memory write.

If this instruction is executed with von neumann Architecture. It requires 4 clock cycles but Harvard and modified Harvard requires less no. of clock cycles.

Von Neuman Architecture:

The General purpose process normally ^{we have this} shapes the architecture * same memory for program and data.

The process perform instruction fetch, decode, Read and execute operations sequentially.

such Architecture in the speed can be increased by pipelining this type of Architecture contains common Address and data BUS, ALU, MAC unit and I/O devices.

This type of Architecture is not suitable for DSP processes.

Harvard Architecture:

The Harvard Architecture has separate memory for program and data there also separate Address and data buses for program and data, because of these separate ^{on chip} memory the internal buses. the speed of execution Architecture is High.

from the fig we can observe that there is PMA Bus [program memory Address] and PMD [program memory Data] Bus separate from program.

Similarly, there is separate data memory. DMA Bus and data memory Address [DMA] Bus.

The DSP includes various Registers, Address generators and ALU.

The PMD Bus is used to get instructions from the program memory and DMA is used to exchange operands and results from data memory.

the instruction code from program memory and operands from data memory can be search simultaneously. Is parallel operation increases the speed.

Modified Harvard Architecture

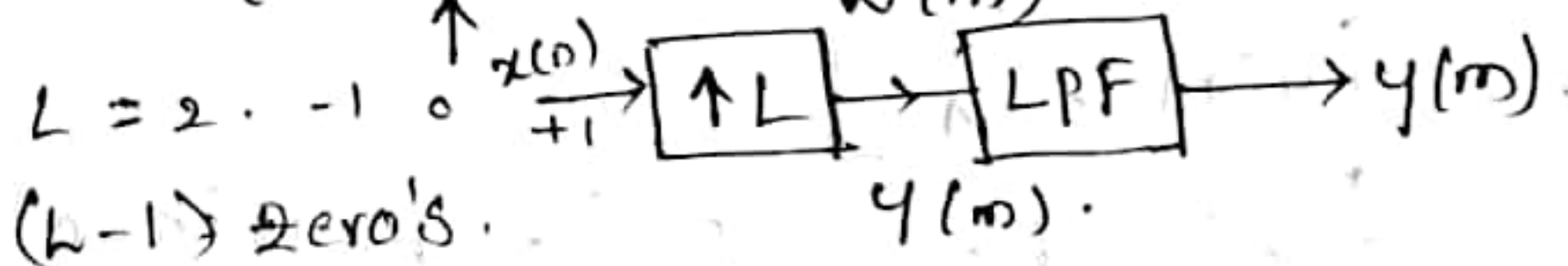
From this architecture one set of bus to access program as well as data memory.

The DMD bus can be use to transfer the data from program memory to data memory. and vice-versa. Normally, the program memory and data memory address are generated by the separated address generators, this modified Harvard architecture. used in several programmable DSP.

14/3/19**
 P) The sequence $x(n) = (0, 2, 4, 6, 8)$ is interpolated using interpolation sequence $b_k = [\frac{1}{2}, 1, \frac{1}{2}]$ and the interpolation factor is '2' to find the interpolated sequence $y(m)$. the 8

6a) Given sequence $x(n) = (0, 2, 4, 6, 8)$.

$$b_k = [\frac{1}{2}, 1, \frac{1}{2}]$$



$$w(m) = (0, 0, 2, 0, 4, 0, 6, 0, 8)$$

$$y(m) = \sum b_k w(m-k)$$

$$y(m) = \sum b_k w(m-k)$$

$$y(0) = \sum b_k w(0-k)$$

$$y(0) = \frac{1}{2} w(0) + 1 w(0) + \frac{1}{2} w(0)$$

$$y(0) = 0$$

$$y(1) = \sum b_k w(1-k)$$

$$y(1) = \frac{1}{2} w(2) + 1 w(1) + \frac{1}{2} w(0)$$

$$= \frac{1}{2} (2) + 1 (0) + \frac{1}{2} (0)$$

$$\boxed{y(1) = 1}$$

$$y(2) = b_K w(2+1) + b_K w(2-0) + b_K w(2-1)$$

$$= b_{-1} w(3) + b_0 w(2) + b_{+1} w(1)$$

$$= \frac{1}{2} (0) + 1 (2) + \frac{1}{2} (0)$$

$$\boxed{y(2) = 2}$$

$$y(3) = b_K w(3+1) + b_K w(3-0) + b_K w(3-1)$$

$$= b_{-1} w(4) + b_0 w(3) + b_{+1} w(2)$$

$$= \frac{1}{2} (4)^2 + 1 (0) + \frac{1}{2} (2)$$

$$\boxed{y(3) = 2+1} \Rightarrow \boxed{y(3) = 3}$$

$$y(4) = b_K w(4+1) + b_K w(4-0) + b_K w(4-1)$$

$$= b_{-1} w(5) + b_0 w(4) + b_{+1} w(3)$$

$$= \frac{1}{2} (0) + 1 (4) + \frac{1}{2} (0)$$

$$\boxed{y(4) = 4}$$

$$y(5) = b_K w(5+1) + b_K w(5-0) + b_K w(5-1)$$

$$= b_{-1} w(6) + b_0 w(5) + b_{+1} w(4)$$

$$= \frac{1}{2} (6)^3 + 1 (0) + \frac{1}{2} (4)^2$$

$$y(5) = 3+2$$

$$\boxed{y(5) = 5}$$

$$y(6) = b_K w(6+1) + b_K w(6-0) + b_K w(6-1)$$

$$= b_{-1} w(7) + b_0 w(6) + b_{+1} w(5)$$

$$= \frac{1}{2} (0) + 1 (6) + \frac{1}{2} (0)$$

$$\boxed{y(6) = 6}$$

$$y(7) = b_K w(7+1) + b_K w(7-0) + b_K w(7-1)$$

$$= b_{-1} w(8) + b_0 w(7) + b_{+1} w(6)$$

$$= \frac{1}{2} (8)^4 + 1(0) + \frac{1}{2} (6)^3$$

$$y(7) = 4 + 3$$

$$\boxed{y(7) = 7}$$

$$y(8) = b_K w(8+1) + b_K w(8-0) + b_K w(8-1)$$

$$= b_{-1} w(9) + b_0 w(8) + b_{+1} w(7)$$

$$= \frac{1}{2} (10) + 1(8) + \frac{1}{2} (0)$$

$$\boxed{y(8) = 8}$$

$$y(m) = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$$

2) The sequence $x(n) = [0, 3, 6, 9]$ is interpolated using interpolation sequence $b_K = [\frac{1}{3}, \frac{2}{3}, 1, \frac{2}{3}, \frac{1}{3}]$ and the interpolation factor is "3" find the interpolated sequence $y(m)$.

sol) Given data.

$$x(n) = [0, 3, 6, 9]$$

$$b_K = [\frac{1}{3}, \frac{2}{3}, 1, \frac{2}{3}, \frac{1}{3}]$$

$$\begin{array}{ccccc} & & \uparrow & & \\ -2 & -1 & 0 & +1 & +2 \end{array}$$

$$L-1 = 0$$

$$3-1 \Rightarrow 2$$

$$w(m) = \{0, 0, 0, 3, 0, 0, 6, 0, 0, 9\}$$

$$y(m) = \sum b_K w(m-K)$$

$$y(m) = b_K w(m+2) + b_K w(m+1) + b_K w(m-0) + b_K w(m+1) + b_K (m-2)$$

$$y(0) = b_{-2} w(2) + b_{-1} w(1) + b_0 w(0) + b_{+1} w(-1) + b_{+2} (-2)$$

$$= \frac{1}{3}(0) + \frac{2}{3}(0) + 1(0) + \frac{2}{3}(0) + \frac{1}{3}(0)$$

$$\boxed{y(0) = 0}$$

$$y(1) = b_{-2} w(3) + b_{-1} w(2) + b_0 w(1) + b_1 w(0) + b_2 w(-1)$$

$$= \frac{1}{3}(3) + \frac{2}{3}(2) + 1(1) + \frac{2}{3}(0) + \frac{1}{3}(-1)$$

$$\boxed{y(1) = 1}$$

$$y(2) = b_{-2} w(4) + b_{-1} w(3) + b_0 w(2) + b_1 w(1) + b_2 w(0)$$

$$= \frac{1}{3}(4) + \frac{2}{3}(3) + 1(2) + \frac{2}{3}(1) + \frac{1}{3}(0)$$

$$\boxed{y(2) = 2}$$

$$y(3) = b_{-2} w(5) + b_{-1} w(4) + b_0 w(3) + b_1 w(2) + b_2 w(1)$$

$$= \frac{1}{3}(5) + \frac{2}{3}(4) + 1(3) + \frac{2}{3}(2) + \frac{1}{3}(1)$$

$$\boxed{y(3) = 3}$$

$$y(4) = b_{-2} w(6) + b_{-1} w(5) + b_0 w(4) + b_1 w(3) + b_2 w(2)$$

$$= \frac{1}{3}(6) + \frac{2}{3}(5) + 1(4) + \frac{2}{3}(3) + \frac{1}{3}(2)$$

$$= 2 + 2$$

$$\boxed{y(4) = 4}$$

$$y(5) = b_{-2} w(7) + b_{-1} w(6) + b_0 w(5) + b_1 w(4) + b_2 w(3)$$

$$= \frac{1}{3}(7) + \frac{2}{3}(6) + 1(5) + \frac{2}{3}(4) + \frac{1}{3}(3)$$

$$= 4 + 1$$

$$\boxed{y(5) = 5}$$

$$y(6) = b_{-2} w(8) + b_{-1} w(7) + b_0 w(6) + b_1 w(5) + b_2 w(4)$$

$$= \frac{1}{3} (0) + \frac{2}{3} (0) + 1 (6) + \frac{2}{3} (0) + \frac{1}{3} (0)$$

$$\boxed{y(6) = 6}$$

$$y(7) = b_{-2} w(9) + b_{-1} w(8) + b_0 w(7) + b_1 w(6) + b_2 w(5)$$

$$= \frac{1}{3} (9)^3 + \frac{2}{3} (0) + 1 (0) + \frac{2}{3} (6^2) + \frac{1}{3} (0)$$

$$y(7) = 3 + 4$$

$$\boxed{y(7) = 7}$$

$$y(8) = b_{-2} w(10) + b_{-1} w(9) + b_0 w(8) + b_1 w(7) + b_2 w(6)$$

$$= \frac{1}{3} (0) + \frac{2}{3} (9)^3 + 1 (0) + \frac{2}{3} (0) + \frac{1}{3} (6^2)$$

$$y(8) = 6 + 2$$

$$\boxed{y(8) = 8}$$

$$y(9) = b_{-2} w(11) + b_{-1} w(10) + b_0 w(9) + b_1 w(8) + b_2 w(7)$$

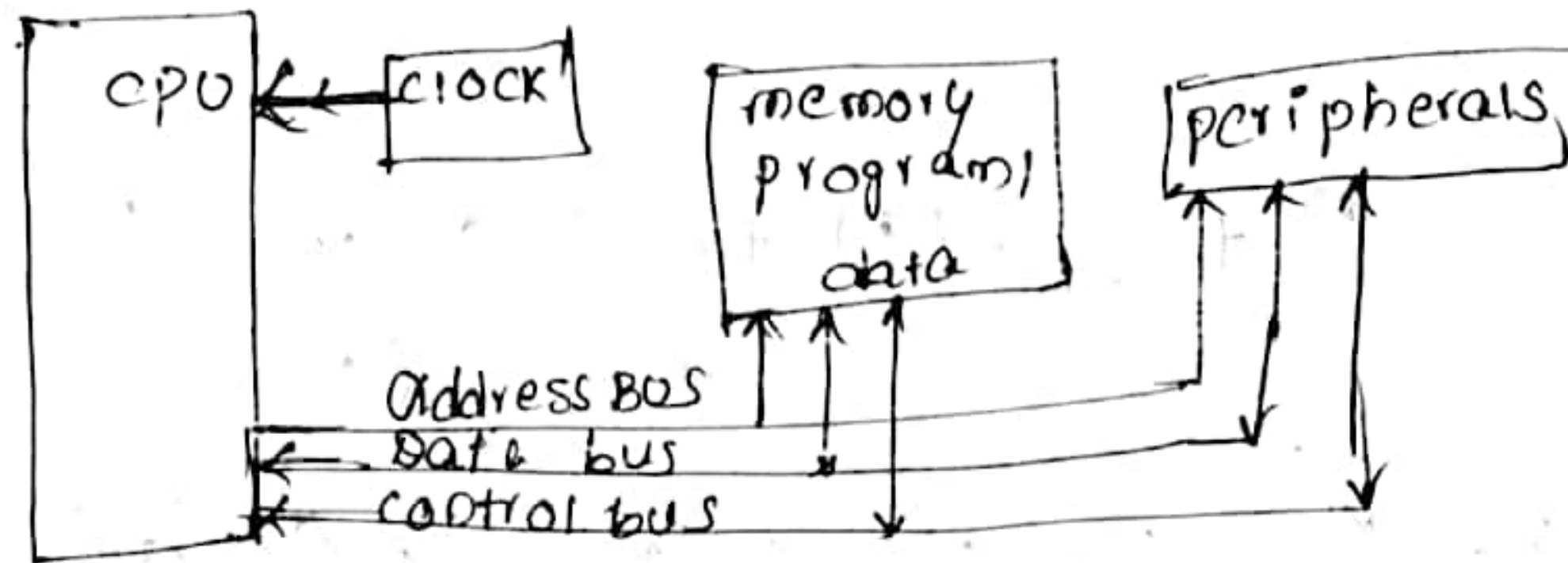
$$= \frac{1}{3} (0) + \frac{2}{3} (0) + 1 (9) + \frac{2}{3} (0) + \frac{1}{3} (0)$$

$$\boxed{y(9) = 9}$$

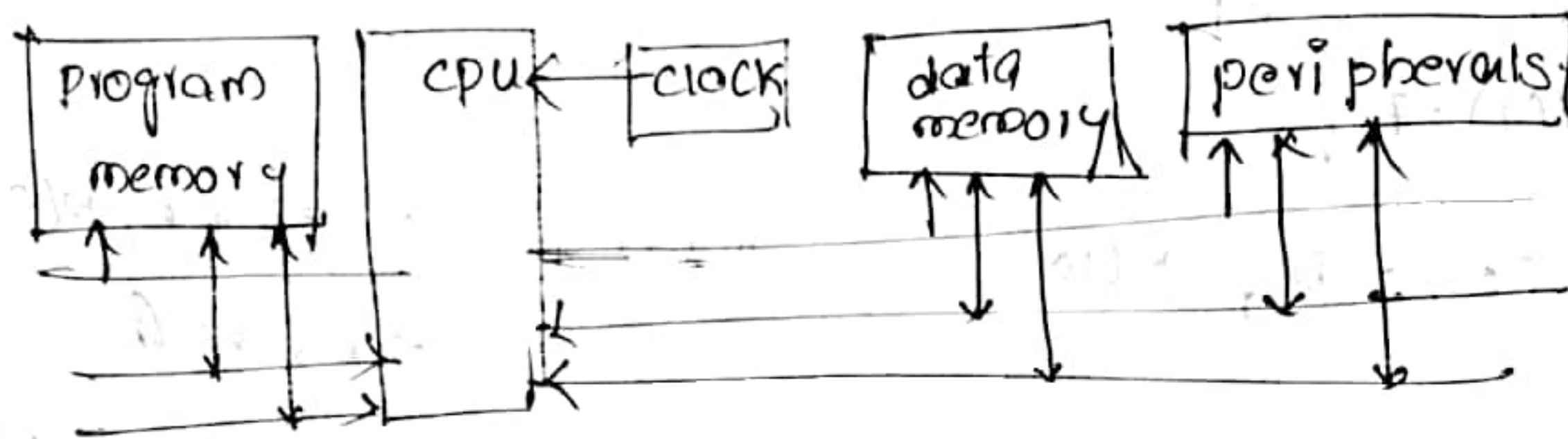
$$y(m) = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

⇒ The pDSP's are divided into
1) General purpose 2) special purpose DSP's.

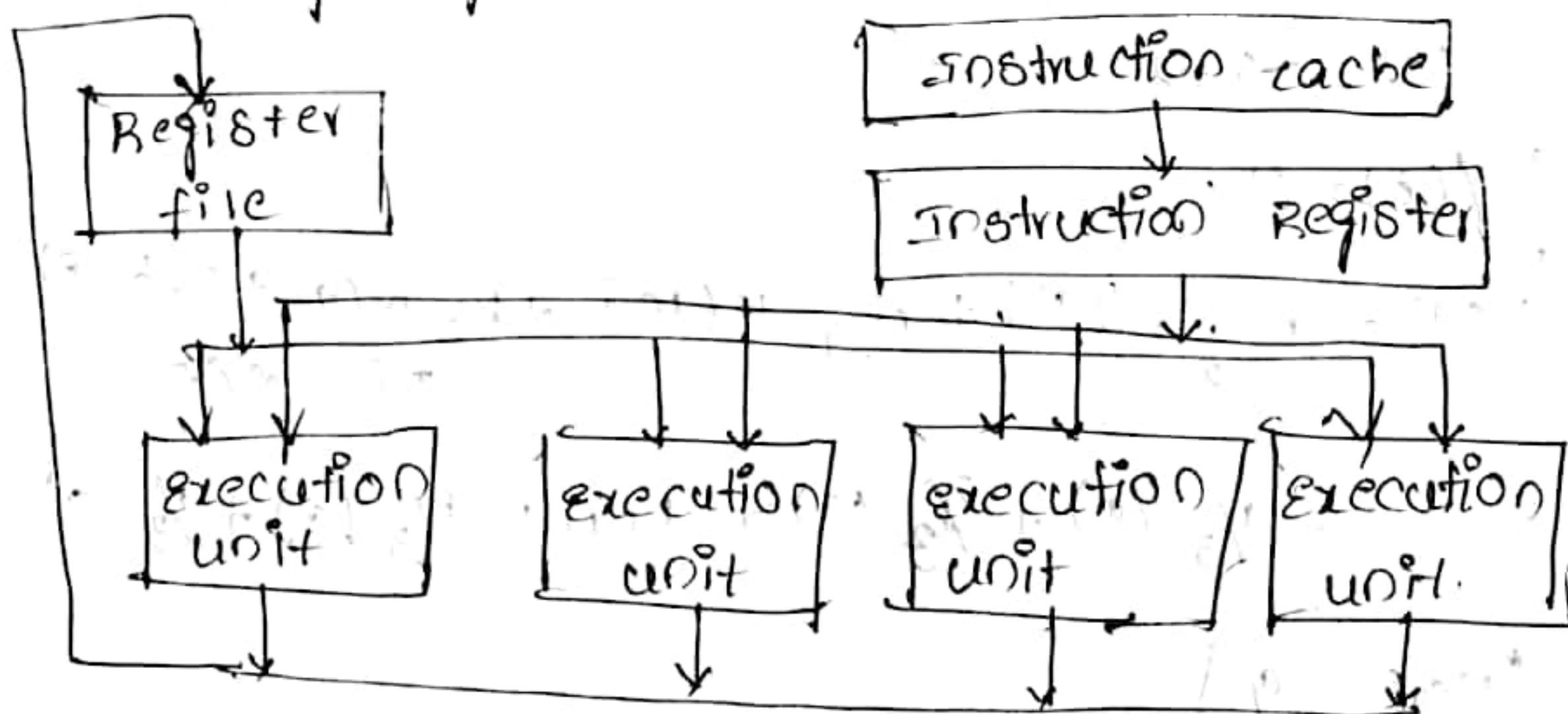
1) Von Neumann Architecture:



2) Harvard Architecture:



3) VLIW (very long instruction word) Architecture:



Architecture of TMS320C50.

It is divided into 4 sub blocks.

1) Bus structure 2) CPU 3) onchip memory 4) on chip peripherals.

peripherals.
Data Bus.

① Bus structure

